PROCEEDINGS OF THE CONFERENCE "ALGEBRA AND LOGIC", MARIBOR 1989
ZBORNIK REFERATOV KONFERENCE "ALGEBRA IN LOGIKA", MARIBOR 1989

FREE FULLY COMMUTATIVE VECTOR VALUED GROUPS
PROSTE POPOLNOMA KOMUTATIVNE GRUPE Z VEKTORSKIMI VREDNOSTMI

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UDK 512.54: 512.6

Descriptors: free commutative monoid, vector valued group, Post theorem

Deskriptorji: prost komutativen monoid, grupa z vektorskimi vrednostmi. Postov izrek

Abstract. In this paper a construction of a free fully commutative vector valued group is given. For this purpose a more general construction of fully commutative vector valued groups, starting from a commutative group with some special properties, is given. As an application of this more general construction, a corresponding Post theorem for fully commutative vector valued groups is obtained.

Izvleček. V članku konstruiramo prosto popolnoma komutativno grupo z vektorskimi vrednostmi. Uporabljena je posplošena konstrukcija popolnoma komutativne grupe z vektorskimi vrednostmi, ki izhaja iz komutativnih grup s posebnimi lastnostmi. Kot primer uporabe te konstrukcije je izpeljan ustrezni Postov izrek za popolnoma komutativne grupe z vektorskimi vrednostmi.

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0. Preliminaries.

Let Q be a nonempty set. Denote by $Q^{(+)}$ the free abelian semigroup with a basis Q. If ρ is a positive integer, let $Q^{(p)}$ be the subset $\{a_1 \ldots a_p \mid a_\nu \epsilon Q\}$ of $Q^{(+)}$, where $a_1 \ldots a_p$ is the product of $a_1 \ldots a_p$ in $Q^{(+)}$. Instead of $a_1 \ldots a_p$ we will use the notation a_1^p, keeping in mind that $a_1^p = b_1^p$ in $Q^{(p)}$ iff $b_1 \ldots b_p$ is a permutation of $a_1 \ldots a_p$. Also, the p–th Cartesian product Q^p could be considered as a subset of the free semigroup Q^+ with a basis Q, identifying it with the subset $\{a_1 \ldots a_p \mid a_\nu \epsilon Q\}$ of Q^+. where $a_1 \ldots a_p$ is the product of $a_1 \ldots a_p$ in Q^+

Let n,m,n–m=k be positive integers. A map $f:Q^{(n)} \to Q^{(m)}$ is called a fully commutative (shortly f.c.) (n,m)–operation on Q, and the ordered pair (Q;f) a f.c. (n,m)–groupoid.

A f.c. (n.m)–groupoid (Q;f) is called f.c. (n,m)–semigroup if for each $x_1^{m+k} \epsilon Q^{(m+k)}$

$$f(f(x_1^n)x_{n+1}^{n+k}) = f(f(x_2^{n+1})\, x_{n+2}^{n+k}\, x_1) \qquad (0.1)$$

The following results (given in [2]) are valid for f.c. (n,m)–semigroups:

Theorem 0.1. (The general fully commutative associative law: GALFC) Let (Q;f) be a f.c. (n,m)–semigroup, n,m,n–m ⩾ 1, and for each s ⩾ 1, $f^{(s)}:Q^{(m+sk)} \to Q(m)$ be defined by

$$f^{(1)}=f,\ f^{(s+1)}\,(a_1^{m+sk}\, b_1^k) = f(f^{(s)}\,(a^{m+sk})\, b_1^k). \qquad (0.2)$$

Then:

(i) For each s ⩾ 1, $(Q;f^{(s)})$ is a f.c. (m+sk,m)–semigroup;

(ii) For each s,t ⩾ 1, $a_\nu, b_\nu \epsilon Q$.

$f^{(t)}\,(f^{(s)}\,(a_1^{m+sk})\, b_1^{tk}) = f^{(s+t)}\,(a_1^{m+sk} b_{tk}^1)$. ■

If (Q;f) is a f.c.(n,m)–semigroup, then we say that the f.c. (m+sk,m)–semigroup $(Q;f^{(s)})$ is derived from (Q;f).

Because of the GALFC, we use the notation $[\ ]:Q^{(n)} \to Q^{(m)}$ instead of $f:Q^{(n)} \to Q^{(m)}$. and $[a_1^{m+sk}]$ instead of $[\ ]^{(s)}\,(a_1^{m+sk})$.

Theorem 0.2. (Post theorem for f.c. v.v. semigroups) If (Q;[]) is a f.c.(m+sk,m)–semigroup, then there exists a f.c. (m+k,m)–semigroup (P; [ ]'), such that Q⊆P, and for every $x_1^{m+sk} \epsilon Q^{(m+sk)}$.

$[x_1^{m+sk}] = [x_1^{m+sk}]'$ ■

We say that a f.c. (n,m)–semigroup is f.c. (n,m)–group if for each $a\epsilon Q^{(k)}$, $b\epsilon Q^{(m)}$, the equation $[ax] = b$ has a solution $x\epsilon Q^{(m)}$.

Since f.c. (n,1)–groups are commutative n–groups, we will always assume that m ⩾ 2. and by a f.c. vector valued group (shortly f.c.v.v. group) we will mean a f.c.(n,m)–group for m ⩾ 2.

Proposition 0.3. There does not exist finite fully commutative vector valued group with more then two elements.■

Let G = (G; $\cdot$) be a commutative group and $Q \subseteq G$ a nonempty subset. We define a family $\{Q_\alpha \mid \alpha \geq 1\}$ of subsets of G by:

$$Q_1 = Q, Q_{\alpha+1} = Q_\alpha \cdot Q, \tag{0.3}$$

where $M \cdot N = \{x \cdot y \mid x \in M, y \in N\}$, for $M, N \subseteq G$. For t, a positive integer, we denote by τ_t the canonical map $Q^{(t)} \to Q_t$ defined by:

$$\tau_t (a_1^t) = a_1 \cdot a_2 \cdot \ldots \cdot a_t. \tag{0.4}$$

Theorem 0.4. Let (G;$\cdot$) be a commutative group, and Q be a nonempty subset of G such that the following conditions are satisfied:

(i) The map τ_m: $Q^{(m)} \to Q_m$ is bijective:

(ii) For each $x \in Q_k$, $Q_m = x \cdot Q_m$ (= x Q_m):

(iii) If $0 \leq i \leq j$ and $Q_{m+1} \cap Q_{m+j} \neq \phi$, then i=j:

(iv) $G = \bigcup_{\alpha=m}^{m+k-1} Q_\alpha$.

Then (Q;[]), where

$$[a_1^{m+k}] = b_1^m \Leftrightarrow \tau_n (a_1^{m+k}) = \tau_m (b_1^m). \tag{0.5}$$

is a f.c. (n,m)–group. Moreover, for each f.c.(n,m)–group (Q; ⟦ ⟧) there exists a commutative group G = (G; $\cdot$) such that $Q \subseteq G$, and the conditions (i)–(iv) are satisfied for Q, and [] defined by (0.5) coincides with ⟦ ⟧. ■

If (Q; []) is a f.c.(n,m)–group, and G=(G; $\cdot$) is a commutative group such that $Q \subseteq G$, and (i)–(iv) are satisfied we say that G is a commutative universal covering group for (Q; []), and denote it by $Q^{(v)}$. In fact. $Q^{(v)}$ is the group given by the presentation $\langle Q; \Lambda \rangle$ in the class of all commutative groups, where $\Lambda = \{a_1 \ldots a_{m+k} = b_1 \ldots b_m \mid [a_1^{m+k}] = b_1^m\}$. If G is a commutative group and Q a subset of G such that (i) holds, we say that Q is m–free in G. If, moreover, Q satisfies (ii). then we say that Q is f.c.(n,m)–subgroup of G, or (n,m)–subgroup of the commutative group G.

1. Fully commutative vector valued groups induced by commutative groups.

We will give here a construction of a f.c.(m+k,m)–group, starting with the given commutative group, by constructing inductively a chain of sets B_α, and a chain of groups G_α, $\alpha \epsilon N$.

Let G=(G; ·) be a commutative group generated by a nonempty set A, A be m–free in G, and n,m, n–m=k be positive integers.

We take G_o to be the given group G.

The norm of an element $x \epsilon G_o$, denoted by $\| \|$, is defined by $\| x \| = 0$, i.e. it is the zero homomorphism from G_o into N.

We will give only the first step of the inductive procedure.

We define B_o, and T_o by:

$B_o := A$, $T_o := G_o$.

Then

$$E_o := \{ x \epsilon G_o \mid (\exists s \geqslant 1)(\ u_1^{m+sk} \epsilon (B_o)^{m+sk})\ x = \tau(u_1^{m+sk})\} \setminus (B_o)^{(m)}.$$

and

$$B_1 := B_o \cup (N_m \times E_o),\ T_1 := G_o \cup (N_m \times E_o)^{(*)},$$

i.e. T_1 is the free product of commutative monoids and $(N_m \times E_o)^{(*)}$ is the free commutative monoid with a basis $(N_m x E_o)$.

Next, we define a norm on the elements of T_1 to be the unique homomorphique extension of the norm defined on T_o and the unique homomorphic extension of the mapping $(i, x) \to 1 + \| x \|$ from $(N_m \times E_o)^{(*)}$ into N.

We say that an element $x \epsilon T$, is reducible if the canonical form of x is $y(1,z) \ldots (m, z)$, otherwise we say that x is reduced. The set of all reduced elements of T_1 is denoted by G_1.

By induction on the norm of the elements of T_1 we define a reduction, i.e. a mapping $\varphi : T_1 \to G_1$. Namely,

(0) $\varphi(x) := x$, for $x \epsilon G_1$.

Let $\varphi(y)$ be well defined for each y such that $\| y \| < \| x \|$.

and

$$\varphi(y) \neq y \Leftrightarrow \| \varphi(y) \| < \| y \|. \qquad (*)$$

If $x = y(1,z_1) \ldots (m, z_1) \ldots (1,z_r) \ldots (m.z_r)$, where y is reduced and $r \geqslant 1$, then

(1) $\varphi(x) := \varphi(yz_1z_2 \ldots z_r)$.

The mapping φ has good properties. Their proof is by induction on the norm of the elements of T_1.

Proposition 1.1. (*a*) φ is a well defined mapping such that for each $x \epsilon T_1$ the condition (*) is fulfilled.

(b) $\varphi(y(1,z)\ldots(m,z)) = \varphi(yz)$.

(c) $\varphi(yz) = \varphi(\varphi(y)z)$.

Proof: (*a*) As $\| yz_1z_2 \ldots z_r \| \leqslant \| x \|$, $\varphi(yz_1z_2 \ldots z_r)$ is defined, and so is $\varphi(x)$. Moreover,

$\| \varphi(x) \| = \| \varphi(yz_1z_2 \ldots z_r) \| \leqslant \| x \|$.

(b) Let y be reducible, and

$y = t(1,z_1)\ldots(m,z_1)\ldots(1,z_r)\ldots(m,z_r)$,

where t is reduced. Then, by the definition of φ and the inductive hypothesis.

$\varphi(t(1,z_1)\ldots(m,z_1)\ldots(1,z_r)\ldots(m,z_r)(1,z)\ldots(m,z)) =$

$\varphi(tz_1z_2\ldots z_r z) = \varphi(t(1,z_1)\ldots(m,z_1)\ldots(m,z_r)z) = \varphi(yz)$.

If $\varphi(x) = x$, then (c) is obviously true. Let $\varphi(x) \neq x$, i. e. $x = t(1,z)\ldots(m,z)$. Then, by (b), we have

$\varphi(xy) = \varphi(t(1,z)\ldots(m,z)y) = \varphi(ty(1,z)\ldots(m,z)) = \varphi(tzy)$.

Now, using the inductive hypothesis, we obtain

$\varphi(tzy) = \varphi(\varphi(tz)y) = \varphi(\varphi(x)y)$. ■

Using the mapping φ, we can define a binary operation * on G_1. Namely,

$x*y := \varphi(xy)$.

Proposition 1.2. $G_1=(G_1;*)$ is a commutative group generated by B_1, and B_1 is m–free in G_1.

Proof: It is obvious that $G_1 = (G_1; *)$ is a commutative monoid, and that G_1 is generated by B_1. Let $(i,x) \epsilon B_1$. Then

$(i,x)^{-1} = x^{-1}(1,x)\ldots(i-1,x)(i+1,x)\ldots(m,x)$.

By the construction of G_1, it follows that $G_o \leqslant G_1$. It remains to prove that B_1 is m–free in G_1.

Let x_ν, $y_\nu \epsilon B_1$, and $x_1 * \dots * x_m = y_1 * \dots * y_m$ in G_1. If x_ν, $y_\nu \epsilon B_o$ for each $\nu \epsilon N_m$, then the claim is true, as b_o is m–free in $G_o \leqslant G_1$.

If both $x_1 \dots x_m$, and $y_1 \dots y_m$ are reducible, then $x_\nu = (\lambda, x)$, $y_\nu = (u, y)$, and we have

$$(1,x) * \dots * (m,x) = (1,y) * \dots * (m, y),$$

i.e. $\varphi(x) = \varphi(y)$, i.e. $x=y$ in G_o. Also, if $x_1 \dots x_m$ is reduced and $y_1 \dots y_m$ is not, we obtain that $x_1 \dots x_m \epsilon E_o$, which contradicts the definition of E_o.

It remains to consider the case when both sides of the equation are reduced, but at least one of them is in $G_1 \setminus G_o$, (thus, both of them are in $G_1 \setminus G_o$).

Let $x_i \dots x_m, y_1, \dots y_m \epsilon G_1 \setminus G_o$, where $2 \leqslant i, j \leqslant m$, and $x_1 \dots x_{i-1}, x_1, \dots x_{j-1} \epsilon G_o$. Then

$$x_i * \dots * x_m = y_j * \dots * y_m,$$

$$x_1 * \dots * x_{i-1} = y_i * \dots * y_{j-i}.$$

But $x_i * \dots * x_m = y_j * \dots * y_m$ imply that i=j, and, then, considering the fact that if a set is m–free in a commutative group then it is r–free for any $1 \leqslant r \leqslant m$, $y_j \dots, y_m$ is a permutation of $x_j, \dots, x_m$, and $y_1, \dots y_{j-1}$ is a permutation of $x_1 \dots, x_{j-1}$. ■

Continuing in this way, we obtain a chain of sets B_α, and a chain of groups G_α. such that B_α generates G_α, and B_α is m–free in G_α. Finally.

$$C := \bigcup_{\alpha \geqslant o} B_\alpha, \quad G := \bigcup_{\alpha \geqslant o} G_\alpha .$$

Proposition 1.3. G is a commutative group generated by C, and C is m–free in G. ■

Using the fact that C is m–free in G, a fully commutative (m+k,m)–operation [] could be defined on C, i.e.

$$[u_1^{m+k}] = v_1^m \Leftrightarrow u_1 * \dots * u_{m+k} = v_1 * \dots * v_m. \tag{1.1}$$

Theorem 1.4. C = (C; []) is a fully commutative (m+k, m)–group.

Proof: It is sufficient to prove that C is an (m+k, m)–subgroup of the commutative group G. As C is m–free in G, we need only to prove that the condition (ii) of Theorem 0.4

is valid for C. Let $u=u_1* \ldots *u_k$, $v=v_1* \ldots *v_m$ be such that $u_\nu, v_\lambda \in C$. Then $uv \in C_m = \{c_1* c_2* \ldots *c_m \mid c_\lambda \in C\}$. Conversely, if u and v are given as above, then there is an element $w \in G$, such that $uw=v$. One can easily prove that $w \in C_m$. ■

Let $Q = (Q; [\![\,]\!])$ be a given f.c. (m+k,m)–group. Using the given construction for obtaining a f.c. (m+1,m)–group, and choosing the group G_o to be the commutative covering group $Q^{(v)}$ of the given f.c. (n,m)–group Q, we obtain a f.c. (m+1,m)–group $C=(C; [\,])$.

It is obvious that $Q \subseteq C$, and

$$[\![a_1^{m+k}]\!] = b_1^m \Leftrightarrow a_1* \ldots *a_{m+k} = b_1* \ldots *b_m \Leftrightarrow [a_1^{m+k}] = b_1^m,$$

for any $a_1, \ldots, a_{m+k} \in Q$.

Thus, we have the following

Theorem 1.5. (The Post Theorem for f.c.v.v.groups). Let $Q = (Q; [\![\,]\!])$ be a f.c. (m+k,m)–group. Then there exists a f.c. (m+1, m)–group $P=(P;[\,])$, such that $Q \subseteq P$, and

$$[\![a_1^{m+k}]\!] = [a_1^{m+k}].$$

for every $a_\nu \in Q$. ■

2. Free fully commutative vector valued groups

Using the construction given in sect. **1** we will give a construction of a free f.c.v.v. group generated by a nonempty set A.

First we define, in the usual way, the notions of subgroups and subgroups generated by a given nonempty subset of the given f.c.v.v. group. Namely, if $Q = (Q; [\,])$ is a f.c. (n.m)–group, then the, nonempty subset $P \subseteq Q$ is called a subgroup of Q iff the following conditions hold:

(i) $a_1^{m+k} \in P^{(m+k)} \Rightarrow [a_1^{m+k}] \in P^{(m)}$.

(ii) $(\forall a \in P^{(k)}, b \in P^{(m)})\ [ax] = b \Rightarrow x \in P^{(m)}$.

Note that a nonempty intersection of subgroups of a f.c. v.v. group is also a subgroup. Thus, we have a natural definition of a f.c.v.v. group generated by a given nonempty set A. We will give a description of the subgroup of a f.c.v.v. group generated by a given subset. Namely, let $Q = (Q; [\,])$ be a f.c. (m+k.m)–group, and A be a nonempty subset of Q. We will, first, define a sequence $H_\alpha(A)$ of subsets of Q in the following way:

(i) $H_o(A) = A$

(ii) $A' = H_o(A) \cup \{a \in Q \mid [x_1^n] = a y_2^m, x_\nu \in H_o(A)\}$;

$A'' = \cup \{\{x_1 \ldots x_m\} \mid [a_1^k x_1^m] = b_1^m, a_\nu, b_\lambda \in H_o(A)\}$,

$H_1(A) = A' \cup A''$:

(iii) $H_{\alpha+1}(A) = H_1(H_\alpha(A))$.

Denote by H(A) the union $\bigcup_{\alpha \geqslant 0} H_\alpha(A)$.

Proposition 2.1. Let Q = (Q; []) be a f.c.(n,m)–group, and A a nonempty subset of Q.

(a) $a \in Q$ is generated by $A \subseteq Q$ iff $a \in H(A)$.

(b) Q is generated by A iff Q=H(A). ■

Let Q = (Q; []) and Q' = (Q' ; []'), be two f.c.(n,m)–groups, and $\varphi: Q \to Q'$ a mapping. We say that φ is a homomorphism iff the following condition holds:

$$[a_1^{m+k}] = b_1^m \Rightarrow [\varphi(a_1) \ldots \varphi(a_{m+k})]' = \varphi(b_1) \ldots \varphi(b_m) \quad (2.1)$$

We say that a f.c.(m+k,m)–group Q = (Q; []) is free with a basis A, if Q is generated by A, and, moreover, every map from A into an arbitrary f.c.(m+k,m)–group P = (P; ⟦ ⟧) has a homomorphic extension $\varphi: Q \to P$.

Let us choose, in the construction given in 1 . G_o to be the free commutative group generated by A. Then

Theorem 2.2. C = (C; []) is a free f.c.(m+k,m)–group with a basis A.

Proof: Let P be a f.c.(m+k,m)–group generated by A. It is clear that each element of G_o is generated by A. Then

$(1, x_1^{m+k}) \ldots (m, x_1^{m+k}) = [x_1^{m+k}]$,

i.e. $(i,x) \in P$, where $x \in E_o \subseteq G_o$. Thus $B_1 \subseteq P$. Let $B_\alpha \subseteq P$, and $x \in E_\alpha$. Then there exist $u_\nu, v_\lambda \in P$, such that $x = u_1 \ldots u_\beta v_1^{-1} \ldots v_\gamma^{-1}$. Let δ be a positive integer such that $\delta + \beta \equiv m$ (mod k). Then

$[u^\delta v_1 \ldots v_\gamma y] = (1, u^\delta u_1 \ldots u_\beta) \ldots (m, u^\delta u_1 \ldots u_\beta)$

has a solution $y = (1,x) \ldots (m,x)$, and $u, u_\nu, v_\gamma, (1, u^\delta u_1 \ldots u_\beta) \in P$. Thus $(i,x) \in P$. i.e. $B_{\alpha+1} \subseteq P$, which imply $C \subseteq P$.

Let Q = (Q; ⟦ ⟧) be a f.c. (m+k,m)–group and $\lambda: A \to Q$ a mapping. As G_o is the free commutative group with a basis A, there exists a unique homomorphic extension ξ_o:

$G_o \to Q^{(v)}$ of λ. Let $\xi_\alpha : G_\alpha \to Q^{(v)}$ be a homomorphic extension of λ. As $T_{\alpha+1}$is the free product of the commutative monoids G_α and $(N_m \times E_\alpha)^{(*)}$, there exists a unique homomorphic extension $\xi'_{\alpha+1}$ of both ξ_α and the homomorphic extension ξ: $(N_m \times E_\alpha)^{(*)} \to Q^{(v)}$ of the mapping $\mu : N_m \times E_\alpha \to Q^{(v)}$, defined by:

$\mu(i, u_1 * \dots * u_{m+k}) = [\![\xi_\alpha(u_1) \dots \xi_\alpha(u_{m+k})]\!]_i$.

Let us note that if $(i,x) \in N_m \times E_\alpha$, then there exist $v_1, \dots, v_{m+k} \in B_\alpha$, such that $x = v_1 * \dots * v_{m+k}$.

Denote by $\xi_{\alpha+1}$ the restriction of $\xi'_{\alpha+1}$ on $G_{\alpha+1}$. To prove that $\xi_{\alpha+1}$ is a homomorphism, it is sufficient to prove that for every $x \in T_{\alpha+1}$, $\xi'_{\alpha+1}(x) = \xi'_{\alpha+1}(\varphi(x))$.

Let $x = x' \ (1,y) \dots (m,y)$. By the properties of the commutative covering group $Q^{(v)}$ and the definition of $\xi'_{\alpha+1}$, it follows that:

$$\xi'_{\alpha+1}(x) = \xi'_{\alpha+1}(x \ (1,y) \dots (m,y)) = \xi'_{\alpha+1}(x')\, \xi'_{\alpha+1}(1,y) \dots \xi'_{\alpha+1}(m,y) =$$
$$= \xi'_{\alpha+1}(x')\, [\![\xi'_{\alpha+1}(y)]\!]_1 \dots [\![\xi'_{\alpha+1}(y)]\!]_m = \xi'_{\alpha+1}(x' \ y).$$

Define a mapping $\xi : G \to Q^{(v)}$ by

$x \in G_\alpha \Rightarrow \xi(x) = \xi_\alpha(x)$.

Then ξ is a homomorphic extension of λ. If we denote the restriction $\xi_{/C}$ by η, then $\eta : C \to Q$ is an $(m+k,m)$–homomorphism. Namely, let $u_1 \dots u_{m+k} \in C$, and let $[u_1^{m+k}] = v_1^m$. Then

$$u_1 * \dots * u_{m+k} = v_1 * \dots * v_m.$$

But, as $\eta = \xi_{/C}$, and ξ is a homomorphism from G into $Q^{(v)}$, we have that

$$\eta(u_1) \dots \eta(u_{m+k}) = \xi(u_1) \dots \xi(u_{m+k}) = \xi(u_1 * \dots * u_{m+k}) =$$
$$= \xi(v_1 * \dots * v_m) = \xi(v_1) \dots \xi(v_m) = \eta(v_1) \dots \eta(v_m).$$

Thus

$[\![\eta(u_1) \dots \eta(u_{m+k})]\!] = \eta(v_1) \dots \eta(v_m)$. ■

In the construction of the homomoprhism η we could have defined that

$\xi(i, u_1 * \dots * u_{m+k}) = [\![\xi(u_1) \dots \xi(u_{m+k})]\!]_j$, for any $1 \leqslant j \leqslant m$. Thus the homomorphic extension with the wanted property is not uniquely defined. This result differes from the results about free algebras. In fact, the following result is true:

Proposition 2.3. Let ξ be an endomorphism of C such that $\xi(a)=a$ for each $a\epsilon A$. Then ξ is an automorphism, and there exist infinitely many such autimorphisms.

Proof: We will prove this proposition by induction on the hierarchy, which is naturaly defined on the elements of the free f.c.(m+k,m)–group C with a basis A, defined in the preivous property. Namely, if $u\epsilon B_{a+1} \setminus B_a$, then the hierarchy $\chi(u)$ is $a+1$.

We define two subsets C_a and D_a of B_{a+1} in the following way:

$d_1, \dots, d_m \epsilon C_a \Leftrightarrow (\exists\, a_1, \dots, a_{m+k} \epsilon B_a)\, [a_1^{m+k}] = d_1^m$,

$t_1 \dots t_m \epsilon D_a \Leftrightarrow (\exists a_1, \dots, a_k, b_1, \dots, b_m \epsilon B_a)\, [a_1^k\, t_1^m\,] = b_1^m$.

Let us note that if $d_j \epsilon C_a \setminus B_\alpha$, then each d_λ , $\lambda=1,2, \dots, m$, is such that $d_\lambda \epsilon C_a \setminus B_a$. Also, considering the construction of C, we obtain that if $t_j \epsilon D_a \setminus (B_a \cup C_a)$, then each $t_\mu \epsilon D_a \setminus (B_a \cup C_a)$, $\mu = 1, \dots, m$, as well. To prove the proposition we need to show that ξ is a permutation on C.

It is clear that ξ is a permutation on B_0. Let ξ be a permutation on B_a and let (1,x) ..., $(m,x)\epsilon C_a \setminus B_a$. Then

$[\xi(a_1) \dots \xi(a_{m+k})\,] = \xi\,(1,x) \dots \xi(m,x)$.

Using the definition of the (m+k,m)–operation we obtain $[\xi(a_1)\ \dots\ \xi(a_{m+k})\ ] =$ $= (1, \xi\,(a_1) \dots \xi(a_{m+k})\,) \dots (m, \xi(a_1) \dots \xi\,(a_{m+k})\,)$.

But, then, $\xi\,(1, x), \dots, \xi\,(m, x)$ is a permutation of

$(1, \xi\,(a_1) \dots \xi\,(a_{m+k})\,), \dots, (m, \xi(a_1) \dots \xi\,(a_{m+k})\,)$.

If $t_1, \dots, t_m \epsilon D_a \setminus (B_a \cup C_a)$, then $t_1 \dots t_m$ is a solution of the equation $[a_1^k\, x_1^m] =$ $= b_1^m$. For some a_ν , $b_\lambda \epsilon B_a$, and

$t_1 \dots t_m = (1, a_1^{-1} \dots a_k^{-1}\, b_1 \dots (m, a_1^{-1} \dots a_k^{-1}\ b_1 \dots b_m)$.

Then $\xi\,(t_1) \dots \xi(t_m)$ is a solution of the equation $[\xi(a_1) \dots \xi(a_k)\, y_1^m] = \xi\,(b_1) \dots$ $\xi(b_m)$.

On the other hand, it has a solution

$(1, \xi(a_1^{-1}\,) \dots \xi\,(a_k^{-1})\, \xi\,(b_1) \dots \xi\,(b_m)\,) \dots$

$\dots (m, \xi\,(a_1^{-1}\ ) \dots \xi\,(a_k^{-1}\ )\, \xi\,(b_1) \dots \xi\,(b_m)\,)$.

As $Q^{(v)}$ is a commutative group, it follows that

$$(1, \xi(a_1^{-1}) \dots \xi(a_k^{-1}) \xi(b_1) \dots \xi(b_m)) \dots$$
$$\dots (m, \xi(a_1^{-1}) \dots \xi(a_k^{-1}) \xi(b_1) \dots \xi(b_m))$$

is a permutation of $\xi(t_1), \dots, \xi(t_m)$. Thus, ξ is a permutation of $B_{a+1} = B_a \cup C_a \cup D_a$. ■

Lemma 2.4. Let Q and Q' be (m+k,m)–groups and $\xi : Q \to Q'$ a homomorphism. Then $\xi(Q)$ is a f.c. (m+k,m)–subgroup of Q' . ■

Proposition 2.5. Let C = (C; []) be the free f.c. (m+k,m)–group constructed in 2.2., and Q = (Q; ⟦ ⟧) be a free f.c.(m+k,m)–group for which A is a basis. Then there exists an isomorphism ψ from C onto Q.

Proof: Using the definition of a free f.c.(m+k,m)–group with a basis A, it follows that the identity mapping from A to A could be extended into a homomorphism ξ from C into Q, and into a homomorphism η from Q into C, and $\eta\xi(a) = a$, for every $a \in A$. By 2.3., then, it follows that ξ is injection. Then, by Lemma 2.4., $\xi(C)$ is subgroup of Q generated by A. Thus $\xi(C) = Q$, and ξ is surjective. ■

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Povzetek

V prvem delu članka navajamo osnovne definicije in izreke o popolnoma komutativnih polgrupah z vektorskimi vrednostmi: splošno popolnoma komutativno asociativno pravilo (Izrek 0.1) in Postov izrek (Izrek 0.2).
V nadaljevanju, izhajajoč iz komutativne grupe z dodatnimi lastnostmi, konstruiramo popolnoma komutativno grupo z vektorskimi vrednostmi (Izrek 1.4) in dokažemo Postov izrek za tovrstno strukturo (Izrek 1.5).
V zadnjem delu članka, kot poseben primer prejšnje splošne konstrukcije, dobimo prosto popolnoma komutativno grupo z vektorskimi vrednostmi (Izrek 2.2).

Prejeto dne 15. 11. 1989