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RECTANGULAR 2-BANDS

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Abstract. In this note we generalize the notion of the rectangular band from binary to ternary operation giving two structure descriptions for the generalized case: first in terms of a ternary groupoid and afterwards in terms of a semigroup.

1. Let S be a 2-groupoid, i.e. a non-empty set S with a ternary operation (...) defined on S. We call S an *anticyclic* 2-groupoid iff: $(xyz) = (yzx) = (zxy) \Longrightarrow x = y = z$.

Let μ be an equivalence relation on S. We call S a *weak associative 2-groupoid with respect to* μ iff the following hold:

$$((xyz)uv) = (x(yzu)v) \Longleftrightarrow z\mu u, \quad (A1)$$

$$((xyz)uv) = (xy(zuv)) \Longleftrightarrow y\mu u, \quad (A2)$$

$$(x(yzu)v) = (xy(zuv)) \Longleftrightarrow y\mu z. \quad (A3)$$

Lemma 1. Every anticyclic weak associative 2-groupoid S is an idempotent 2-groupoid.

Proof. For every $a \in S$ we have that

$$((aaa)aa) = (a(aaa)a) = (aa(aaa)),$$

which implies that $(aaa) = a$. ||

Let us define the equivalence relation μ on S as follows:

$$x\mu y \Longleftrightarrow (\forall a,b \in S) \quad (axb) = (ayb).$$

An anticyclic weak associative 2-groupoid S is said to be a *rectangular 2-band* if the equivalence relation μ is defined as above. From now on S will stand for a rectangular 2-band.

From the definition of μ and Lemma 1 it follows that:

Lemma 2. If $x\mu y$, $x,y \in S$, then $(xyx) = x$. ||

<u>Example</u>. Let A, B and C be non-empty sets and let us define a ternary operation $[\ldots]$ on $S = A \times B \times C$ by

$$[(a_1,b_1,c_1)(a_2,b_2,c_2)(a_3,b_3,c_3)] = (a_1,b_2,c_3),$$

$a_j \in A$, $b_j \in B$, $c_j \in C$. It is easily seen that S is a rectangular 2-band; we shall denote this rectangular 2-band by $S = [A,B,C]_{pr}$.

<u>Lemma 3</u>. Each equivalence class of S (mod μ) is an anticommutative 2-semigroup.

<u>Proof</u>. Let S' be an equivalence class of S modulo μ and let $x,y,z \in S'$. Let $u,v \in S$. Then we have that:

$$(u(xyz)v) = ((uxy)zv) = ((uyy)zv) = (uy(yzv)) = (uy(yyv)) = \\ = (u(yyy)v) = (uyv)$$

which shows that $(xyz)\mu y$, i.e. $(xyz) \in S'$ meaning that S' is a 2-subgroupoid of S.

Since S' is, obviously associative, i.e. 2-semigroup, it remains to prove the anticommutativity of S': if for some $y \in S'$, $(xyz) = (zyx)$, then $x = z$ (see [2]). Really, we have that

$$x = (xxx) = (xyx) = (x(yzy)x) = ((xyz)yx) = ((zyx)yx) = \\ = (zy(xyx)) = (zyx) = (z(yzy)x) = (zy(zyx)) = \\ = (zy(xyz)) = (z(yxy)z) = (zyz) = z. \ ||$$

<u>Lemma 4</u>. For every $x,y,z \in S$, $(xyz)\mu y$.

<u>Proof</u>. Let $(xxz) = u$; then,

$$(xxu) = (xx(xxz)) = ((xxx)xz) = (xxz) = u,$$

and from $(x(xxu)x) = (xux) = ((xxx)ux)$, accroding to (A1), it follows that $u\mu x$, i.e. $(xxz)\mu x$. Similarly we get that $(xyy)\mu y$.

If $v\mu y$, because of $(yyz)\mu y$, from

$$(vyz) = (v(yyy)z) = (vy(yyz))$$

if tollows that $(vyz)\mu y$ since v,y and (yyz) all belong to the same equivalence class (Lemma 3).

Finally,

$$(xyz) = (x(yyy)z) = ((xyy)yz) = (vyz)\mu y,$$

since $(xyy) = v\mu y$. ||

Theorem 1. Every two equivalence classes of S (mod μ) are isomorphic.

Proof. Let S^a and S^b be two equivalence classes with $a \in S^a$, $b \in S^b$. If we put $f_{ab}(x) = (xbx)$, $x \in S^a$, we have that $f_{ab}(x) \in S^b$ (Lemma 4), and, so, f_{ab} is a mapping from S^a to S^b.

Let $f_{ab}(x) = f_{ab}(y)$, $x,y \in S^a$; let $(xbx) = c = (yby)$. We have that

$$(cxc) = ((xbx)xc) = (x(bxx)c) = (xxc) =$$
$$= (xx(xbx)) = (x(xxb)x) = (xxx) = x,$$

since $(bxx),(xxb)\mu x$. So,

$$(cxc) = x, \text{ and similarily, } (ycy) = y \qquad (1)$$

Since $b\mu c$, we have that

$$(ycy) = c,\ (xcx) = c. \qquad (2)$$

Now, taking into account (1) and (2) we get:

$$x = (cxc) = ((ycy)xc) = (y(cyx)c) = (yyc) = (yy(ycy)) =$$
$$= (y(yyc)y) = (yyy) = y.$$

Thus, f_{ab} is an injection.

If $z \in S^b$ and if we put $x = (zaz)$ then $x \in S^a$ and

$$f_{ab}(x) = (xbx) = ((zaz)bx) = (z(azb)x) = (zzx) =$$
$$= (zz(zaz)) = (z(zza)z) = (zzz) = z$$

which shows that f_{ab} is a surjection, also.

Finally, let $x,y,z \in S^a$. Then,

$$(f_{ab}(x)f_{ab}(y)f_{ab}(z)) = ((xbx)(yby)(zbz)) = ((xbx)b(zbz)$$
$$= (xb(xb(zbz))) = (xb((xbz)bz)) =$$
$$= (x(b(xbz)b)z) = (x(bbb)z) = (xbz).$$

On the other hand if we put $(xbx) = u$, $(zbz) = v$, then according to (1) and (2) we have that $x = (uxu)$, $z = (vzv)$ where $u,v \in S^b$ which, according to Lemma 4, implies $(zvb),(buv),(yvb)$ and (buy) all belong to S^b. Thus,

$$\begin{aligned} f_{ab}(xyz) &= ((xyz)b(xyz)) = ((xy(vzv))b(xyz) = \\ &= (((xyv)zv)b(xyz)) = ((xyv)(zvb)(xyz)) = \\ &= ((xyv)b(xyz)) = ((xyv)b((uxu)yz)) = ((xyv)b(ux(uyz))) = \\ &= ((xyv)(byx)(uyz)) = ((xyv)b(uyz)) = (x(yvb)(uyz)) = \\ &= (xb(uyz)) = (x(buy)z) = (xbz). \end{aligned}$$

So, $f_{ab}(xyz) = (f_{ab}(x)f_{ab}(y)f_{ab}(z))$. ||

<u>Lemma 5</u>. Let $x \in S^a$, $y \in S^b$, $z \in S^c$. Then,

$$(xyz) = (f_{ab}(x)yf_{cb}(z)).$$

<u>Proof</u>. Since $(xyz)\mu b$ and $b\mu y$,

$$\begin{aligned} (f_{ab}(x)yf_{cb}(z)) &= ((xbx)y(zbz)) = (xb(xy(zbz))) = (xb((xyz)bz) = \\ &= (x(b(xyz)b)z) = (x(bbb)z) = (xbz) = (xyz). \text{ ||} \end{aligned}$$

<u>Theorem 2</u>. A 2-groupoid S is a rectangular 2-band iff there exist non-empty sets, A,B and C such that $S \cong [A,B,C]_{pr}$.

<u>Proof</u>. For every anticommutative 2-semigroup G there exists a rectangular band G(o) such that for every $x,y,z \in G$, $(xyz) = xoy$ (see [2]). On the other hand (see, for example [1]), every rectangular band is isomorphic to a direct product AxC of a left-zero semigroup A and a right-zero semigroup C. Thus, every anticommutative 2-semigroup G is 2-isomorphic to a rectangular band AxC, i.e. there is a bijection $g: G \to AxC$ such that, for every $x,y,z \in G$, $g(xyz) = g(x)og(z) = g(x)og(y)og(z)$.

Let us return, now, to the rectangular 2-band S; Let S^{b_1} and S^{b_2} be two equivalence classes of S (S^{b_1} and S^{b_2} are anticommutative 2-semigroups according to Lemma 3). Hence, there is a 2-isomorphism $h_1: S^{b_1} \to AxC$, where A and C are, respectively, a left-zero and a right-zero semigroups. Let $f_{12}: S^{b_1} \to S^{b_2}$ be the isomorphism of Theorem 1. Then $h_2 = h_1 f_{12}^{-1}: S^{b_2} \to A\times C$ will be a 2-isomorphism such that: if $x \in S^{b_1}$, $h_1(x) = (a,c)$ and if $y = f_{12}(x)$, $h_2(y) = h_1(f_{12}^{-1}(y)) = h_1(x) = (a,c)$.

Let B be the index set for the family of all equivalence classes of S (mod μ):

$$S = \bigcup\{S^b \mid b \in B\}$$

and let us define a ternary operation $[\ldots]$ on $\overline{S} = A\times B\times C$ as in the Example. Let $\phi: S \to \overline{S}$ be a mapping defined in the following way: if $x \in S^{b_1}$ and $h_1(x) = (a,c)$, then $\phi(x) = (a,b_1,c)$. It is obvious that ϕ is a bijection. If $x,y,z \in S$, where $x \in S^{b_1}$, $y \in S^{b_2}$, $z \in S^{b_3}$ and $h_1(x) = (a_1,c_1)$, $h_2(y) = (a_2,b_2)$, $h_3(z) = (a_3,c_3)$, then $\phi(x) = (a_1,b_1,c_1)$, $\phi(y) = (a_2,b_2,c_2)$, $\phi(z) = (a_3,b_3,c_3)$ and, in $\overline{S}$, we get

$$[\phi(x)\phi(y)\phi(z)] = [(a_1,b_1,c_1)(a_2,b_2,c_2)(a_3,b_3,c_3)] = $$
$$= (a_1,b_2,c_3).$$

On the other hand, according to Lemma 5, we have that

$$(xyz) = (f_{12}(x)yf_{32}(z) \in S^{b_2}, \text{ and}$$

$$h_2(xyz) = (h_2f_{12})(x) \circ h_2(y) \circ (h_2f_{32})(z) =$$
$$= (h_2f_{12})(x) \circ (h_2f_{32})(z) \text{ in } A\times C.$$

Since $h_2 = h_1 f_{12}^{-1}$, or $h_1 = h_2 f_{12}$, as we proved above, and similarly $h_3 = h_2 f_{32}$, we have that

$$(h_2f_{12})(x) = h_1(x) = (a_1,c_1),\ (h_2f_{32})(z) = h_3(z) = (a_3,c_3),$$

and, then,

$$h_2(xyz) = (a_1,c_1)\circ(a_3,c_3) = (a_1,c_3),$$

which, according to the definition of ϕ means that

$$\phi(xyz) = (a_1,b_2,c_3),$$

and, therefore, $\phi(xyz) = [\phi(x)\phi(y)\phi(z)]$ which proves that $\phi: S \to \overline{S}$ is an isomorphism. The converse is obvious. ||

2. Here we shall give another structure description for a rectangular 2-band S.

Lemma 6. Let $f_{12}: S^{b_1} \to S^{b_2}$ and $f_{23}: S^{b_2} \to S^{b_3}$ be the isomorphisms defined in Theorem 1. Then $f_{23}f_{12} = f_{13}$ and $f_{12}f_{21}=\varepsilon_{b_2}$.

Proof. Let $f_{12}(x) = (xbx) = y$, $x \in S^{b_1}$, $b,y \in S^{b_2}$, $f_{23}(y) = (ycy) = z$, $c,z \in S^{b_3}$. Then $(xbx) = y$ implies $(yxy) = x$, $(xyx) = y$, and $(ycy) = z$ implies $(zyz) = y$, $(yzy) = z$ (Theorem 1). Now,

$$\begin{aligned} f_{23}f_{12}(x) &= (ycy) = ((xbx)cy) = ((xb(yxy))cy) = ((x(byx)y)cy) = \\ &= ((xyy)cy) = ((xy(zyz))cy) = (((xyz)yz)cy) = \\ &= ((xyz)(yzc)y) = ((xyz)cy) = (x(yzc)y) = (xcy) = \\ &= (xc(xbx)) = (xc((yxy)bx) = (xc(y(xyb)x)) = \\ &= (xc(yyx)) = (xc((zyz)yx)) = (xc(zy(zyx))) = \\ &= (x(czy)(zyx)) = (xc(zyx)) = (x(czy)x) = (xcx) = f_{13}(x), \end{aligned}$$

i.e. $f_{23}f_{12} = f_{13}$. From this is follows that $f_{12}f_{21}(y) = (yby) = y$, i.e. $f_{12}f_{21} = \varepsilon_{b_2}$. ||

Let us observe that the isomorphisms $f_{ij}: S^i \to S^j$, $i,j \in B$, between the anticommutative 2-semigroups can be considered as isomorphisms between the corresponding rectangular bands $S^i(o_i)$ and $S^j(o_j)$ (see proof of Theorem 2): if $x,y,z \in S^i$,

$$\begin{aligned} f_{ij}(x o_i y) &= f_{ij}(xyz) = (f_{ij}(x) f_{ij}(y) f_{ij}(z)) = \\ &= f_{ij}(x) o_j f_{ij}(z). \end{aligned}$$

Thus, for every rectangular 2-band S there exist a family of rectangular bands $R = \{S^j(o_j) \mid j \in B\}$ and a family $F = \{f_{ij}: S^i \to S^j \mid i,j \in B\}$ of isomorphisms such that $f_{jk}f_{ij} = f_{ik}$, $f_{ij}f_{ji} = \varepsilon_j$ (ε_j - identity on S^j) and, according to Lemma 5,

$$(xyz) = (f_{ij}(x) y f_{kj}(z) = f_{ij}(x) o_j f_{kj}(z).$$

Conversely, let $R = \{S^j(o_j) \mid j \in B\}$ be a family of mutually isomorphic rectangular bands and $F = \{f_{ij}: S^i \to S^j \mid i,j \in B\}$ a family of isomorphisms such that $f_{jk}f_{ij} = f_{ik}$ and $f_{ij}f_{ji} = \varepsilon_j$. Let $S = \bigcup\{S^j \mid j \in B\}$ and define a ternary operation on S as follows:

if $x\in S^i$, $y\in S^j$, $z\in S^k$ then $(xyz) = f_{ij}(x)\circ_j f_{kj}(z)$. We shall denote this algebraic structure on S by $S = [R,F,(\ldots)]$. Let μ be the equivalence in S which corresponds to the partition R of S. Now,

(i) Let $x\in S^i$, $y\in S^j$, $z,u\in S^k$, $v\in S^m$ (hence $z\mu u$). Then

$$((xyz)uv) = ((f_{ij}(x)\circ_j f_{kj}(z))uv) = f_{jk}(f_{ij}(x)\circ_j f_{kj}(z))\circ_k f_{mk}(v) =$$
$$= (f_{jk}f_{ij}(x)\circ_k f_{jk}f_{kj}(z))\circ_k f_{mk}(v) =$$
$$= (f_{ik}(x)\circ_k z)\circ_k f_{mk}(v) = f_{ik}(x)\circ_k z\circ_k f_{mk}(v) = f_{ik}(x)\circ_k f_{mk}(v),$$

and, from the other hand,

$$(x(yzu)v) = (x(f_{jk}(y)\circ_k u)v) = f_{ik}(x)\circ_k f_{mk}(v), \text{ hence}$$
$$((xyz)uv) = (x(yzu)v).$$

Conversely, if $((xyz)uv) = (x(yzu)v)$ and $z\in S^k$, $u\in S^{k'}$ (the other elements as before), repeating the above calculations we get that

$$((xyz)uv) = f_{ik'}(x)\circ_{k'} f_{mk'}(v)\in S^{k'}, \quad (x(yzu)v) = f_{ik}(x)\circ_k f_{mk}(v)\in S^k$$

which implies $k'=k$, i.e. $z\mu u$.

(ii) If $x\in S^i$, $y,u\in S^j$, $z\in S^k$, $v\in S^m$, we get

$$((xyz)uv) = ((f_{ij}(x)\circ_j f_{kj}(z))uv) = f_{jj}(f_{ij}(x)\circ_j f_{kj}(z))\circ_j f_{mj}(v) =$$
$$= f_{ij}(x)\circ_j f_{kj}(z)\circ_j f_{mj}(v) = f_{ij}(x)\circ_j f_{mj}(v), \text{ and}$$
$$(xy(zuv)) = f_{ij}(x)\circ_j (f_{kj}(z)\circ_j f_{mj}(v)) = f_{ij}(x)\circ_j f_{mj}(v)$$

so that $((xyz)uv) = (xy(zuv))$.

For the converse part, if $x\in S^i$, $y\in S^j$, $u\in S^{j'}$, $z\in S^k$, $v\in S^m$ we have that $((xyz)uv) = f_{ij'}(x)\circ_{j'} f_{mj'}(v)$, $(xy(zuv)) = f_{ij}(x)\circ_j f_{mj}(v)$ and $((xyz)uv) = (xy(zuv))$ implies $j=j'$, i.e. $y\mu u$.

(iii) Let $x\in S^i$, $y,z\in S^j$, $u\in S^k$, $v\in S^m$; then,

$$(x(yzu)v) = (x(f_{jj}(y)\circ_j f_{kj}(u))v) = f_{ij}(x)\circ_j f_{mj}(v),$$
$$(xy(zuv)) = (xy(f_{jk}(z)\circ_k f_{mk}(v)) = f_{ij}(x)\circ_j f_{kj}(f_{jk}(z)\circ_k f_{mk}(v)) =$$
$$= f_{ij}(x)\circ_j (f_{kj}f_{jk}(z)\circ_j f_{kj}f_{mk}(v)) =$$
$$= f_{ij}(x)\circ_j (f_{jj}(z)\circ_j f_{mj}(v)) = f_{ij}(x)\circ_j f_{mj}(v),$$

so that, $(x(yzu)v) = (xy(zuv))$.

On the other hand, if $x \in S^i$, $y \in S^j$, $z \in S^{j'}$, $u \in S^k$, $v \in S^m$, then $(x(yzu)v) = f_{ij'}(x) o_{j'} f_{mj'}(v)$, $(xy(zuv)) = f_{ij}(x) o_j f_{mj}(v)$ and $j=j'$, or $y \cup z$, if $(x(yzu)v) = (xy(zuv))$.

Now we shall prove that $S = [R,F,(\ldots)]$ is anticyclic. Let $x \in S^i$, $y \in S^j$, $z \in S^k$ and $(xyz) = (yzx) = (zxy)$. This means that $f_{ij}(x) o_j f_{kj}(z) = f_{jk}(y) o_k f_{ik}(x) = f_{ki}(z) o_i f_{ji}(y)$ which is possible only if $i=j=k$ and we have that $x o_i z = y o_i x = z o_i y$. From $x o_i z = y o_i x$ it follows that $x o_i z o_i x = y o_i x o_i x$, i.e. $x = y o_i x$, and, since $y o_i x = z o_i y$, $x = z o_i y$ and then, $x o_i y = z o_i y o_i y = z o_i y$. Since $z o_i y = y o_i x$ we finally get $x o_i y = y o_i x$ which, because of the anticommutativity of S^i implies $x=y$. Similarly, we can conclude that $y=z$.

From the above considerations we have that:

<u>Theorem 3</u>. i) Let S be a rectangular 2-band. Then there exist a family $R = \{S^j \mid j \in B\}$ of anticommutative, disjoint and mutually isomorphic semigroups and a family $F = \{f_{ij}: S^i \to S^j \mid i,j \in B\}$, of isomorphisms where $f_{ij} f_{ki} = f_{kj}$, $f_{ij} f_{ji} = \varepsilon_j$, such that: if $x \in S^i$, $y \in S^j$, $z \in S^k$, then

$$(xyz) = f_{ij}(x) o_j f_{kj}(z), \tag{3}$$

where "o_j" is the operation in S^j.

Conversely, let $R = \{S^j(o_j) \mid j \in B\}$ be a family of disjoint and mutually isomorphic anticommutative semigroups (rectangular bands), and $F = \{f_{ij}: S^i \to S^j \mid i,j \in B\}$ a family of isomorphisms such that $f_{ij} f_{ki} = f_{kj}$, $f_{ij} f_{ji} = \varepsilon_j$. If we define in $S = \bigcup \{S^j \mid j \in B\}$ a ternary operation with (3), then S will turn out to be a rectangular 2-band. ||

R E F E R E N C E S

[1] Ljapin S.E.: Semigroups, Fizmatgiz, Moscow, 1960 (in Russian)

[2] Trpenovski B.L.: On anticommutative n-semigroups, God. Zb. EMF, Skopje, T.1, 1967, 33-35 (in Macedonian)

ПРАВОАГОЛНИ 2-ЛЕНТИ

Б. Трпеновски

Р е з и м е

Главниот резултат на работава е содржан во теоремите 2 и 3 со кои се дава опис на структурата на правоаголните 2-ленти, воведени како обопштување на правоаголните ленти (антикомутативните полугрупи од идемпотенти).