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ON IDEALS IN REGULAR n-SEMIGROUPS

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In this paper we introduce a new definition of regular n-semigroups and prove some theorems about special elements in that structure. Most attention has been paid to some idealtheoretic aspects. We prove also that every ideal of a regular and commutative (m, n)-ring is a radical ideal.

1. Regular n-semigroups

Let f be an n-ary operation in a set G. Let us denote

$$f\,(x_1, \ldots, x_n) = f\,(x_1^n),$$

$$f\,(x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+s}, x_{k+s+1}, \ldots, x_n) = f\,(x_1^k, \overset{(s)}{x}, x_{k+s+1}^n)$$

whenever $x_{k+1} = x_{k+2} = \ldots = x_{k+s} = x$ (x_i^j is the empty symbol for $j < i$ also $\overset{(0)}{x}$ is the empty symbol).

We shall use terminology and notations of papers [4] and [2].

An n-semigroup $< G, f >$ is called regular [6], *if*

$$(1)\ (\forall\ a \in G)\ (\exists\ x_2, \ldots, x_{2n-2} \in G)\ f_{(2)}\,(a, x_2^{n-}\ a) = a.$$

An ordinary regular semigroup is a special case of a regural n-semigroup, namely for $n=2$.

In the sequel we consider only n-semigroups (n-groups) where $n > 2$.

It is clear that the condition (1) implies

(2) $(\forall a \in G)$ $(\exists z_2, z_3, \ldots, z_{n-1} \in G)$ $f(a, z_2^{n-1}, a) = a.$

A non-empty subset $B_j \subset G$ is called an j-ideal of the n-semigroup $< G, f >$, if $x_j \in B_j$ implies $f(x_1, \ldots, x_j, \ldots, x_n) \in B_j$ for all $x_1, \ldots,$ $\ldots, x_{j-1}, x_{j+1}, \ldots, x_n \in G$ and fixed $1 \leqslant j \leqslant n$.

An j-ideal for each $j = 1, 2, \ldots, n$ is simply called an *ideal.*
The set

$$(a)_j = \{f(x_1^{j-1}, a, x_{j+1}^n) : x_i \in G\} \cup \{a\}$$

is called *a principial j-ideal* generatep by an element $a \in G$.

Theorem 1. In an n-semigroup $< G, f >$ the next conditions are equivalent:

(i) $< G, f >$ is regular,

(ii) $\bigcap_{j=1}^{n} B_j \subset f(B_1, B_{n-1}, B_{n-2}, \ldots, B_2, B_n)$ for all j-ideals B_j,

(iii) $\bigcap_{j=1}^{n} (a_j)_j \subset f((a_1)_1, (a_2)_{n-1}, \ldots, (a_{n-1})_2, (a_n)_n)$

for all $a_1, a_2, \ldots, a_n \in G$,

(iv) $\bigcap_{j=1}^{n} (a)_j \subset f((a)_1, (a)_{n-1}, \ldots, (a)_3, (a)_2, (a)_n)$ for each $a \in G$.

Proof. (i) $\rightarrow$ (ii) If $a \in \bigcap_{j=1}^{n} B_j$ and $< G, f >$ is regular, then there exists $z_2, \ldots, z_{n-1} \in G$ such that

$$a = f(a, z_2^{n-1}, a) = f(f(a, z_2^{n-1}, a), z_2^{n-1}, a) = \ldots =$$

$$= f(f(a, z_2^{n-1}, a), f(z_2^{n-1}, a, z_2), f(z_3^{n-1}, a\ z_2^3), \ldots,$$

$$\ldots, f(z_{n-1}, a, z_2^{n-1}), f(a\ z_2^{n-1}, a))$$

$$\in f(B_1, B_{n-1}, B_{n-2}, \ldots, B_3, B_2, B_n),$$

since $f(a, z_2^{n-1}, a) \in B_1 \cap B_n$ and $f(z_i^{n-1}, a, z_2^i) \in B_{n-i+1}$.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) Obvious.

(iv) $\Rightarrow$ (i) For all $a \in G$ We have $a \in \bigcap_{j=1}^{n} (a)_j$. Hence, for each $a \in G$ there exists elements $b_i \in (a)_i$ such that

(3) $a = f(b_1, b_{n-1}, b_{n-2}, \ldots, b_2, b_2, b_n)$.

If $b_i \in (a)_i$, then $b_i = a$ or

(4) $b_i = f(x_{i1}^{ii-1}, a, x_{ii+1}^{in})$ for some $x_{ij} \in G$.

But if $b_i = a$, then

(5) $b_i = a = f(b_1, b_{n-1}, \ldots, b_{i+1}, a, b_{i-1}, \ldots, b_2, b_n)$.

In view of associative law and (3), (4), (5) we have

$$a = f(f(a, x_{12}^{1n}), \ldots, f(x_{n1}^{nn-1}, a)) = f_{(2)}(a, x_{12}^{1n-1}, f_{(n-1)}(x_{1n}, \ldots, \ldots, x_{n1}), x_{n2}^{nn-1}, a),$$

which completes the proof of our Theorem. ∎

In this same manner as (i) $\rightarrow$ (ii) we prove:

COROLLARY. If $< G, f >$ is a regular n-semigroup, then $\bigcap_{j=1}^{n} B_j \subset f(B_n, B_{n-1}, B_{n-2}, \ldots, B_2, B_1,)$ for all j-ideals B_j. ∎

Theorem 2. An n-semigroup $< G, f >$ is regurlar if and only if every ideal of G is an idempotent ideal, i. e. $B = f(B, \ldots, B)$ for every ideal B.

Proof, Puting $B = B_1 = B_2 = \ldots = B_n$ in Theorem 1 (ii), we have $B \subset f(B, \ldots, B)$. But B is an ideal, hence f $(B, \ldots, B) \subset B$. Therefore regularity implies $B = f(B, \ldots, B)$ for all ideals B.

Conversely, if all B_j are j-ideals in an n-semigroup $< G, f >$, then $\bigcap_{j=1}^{n} B_j$ is an ideal su h that $\bigcap_{j=1}^{n} B_j \subset B_j$ for each $j = 1, 2, \ldots, n$.

This imlies that

$$\bigcap_{j=1}^{n} B_j = f\left(\bigcap_{j=1}^{n} B_j, \ldots, \bigcap_{j=1}^{n} B_j\right) \subset f\ (B_1, B_{n-1}, \ldots, B_2, B_n)$$

i. e. $< G, f >$ is regular. ∎

An n-semigroup $< G. f >$ is called regular in the sense of Sicson, if

$$(6)\begin{cases} (\forall a \in G)\ (\forall\ i, j=1, 2, \ldots, n)\ (\exists x_{ij} \in G) \\ a = f\,(f\ (a,\ x_{12}^{1n}),\ f\ (x_{21},\ a,\ x_{23}^{2n}), \ldots,\ f\ (x_{n1}^{nn-1},\ a)). \end{cases}$$

Theorem 3. (Sioson [7]. If $< G,f >$ is an n-semigroup, then next conditions are equivalent;

(i) $< G, f >$ satisfies (6),

(ii) $f\ (B_1,\ B_2, \ldots,\ B_n,) = \bigcap_{j=1}^{n} B_j$ for all j-ideals B_j,

(iii) $f\,((a_1)_1,\ (a_2)_2, \ldots,\ (a_n)_n) = \bigcap_{j=1}^{n} (a_j)_j$ for all $a_1, \ldots,\ a_n \in G$,

(iv $f\,((a)_1,\ (a)_2, \ldots,\ (a)_n) = \bigcap_{j=1}^{n} (a)_j$ for all $a \in G$,

(v) every ideal is idempotent. ∎

F. M. Sioson proved the condition (v) only for commutative *n-semi*-groups. It is also true for *non*-commutative n-semigroups.
The proof is anologous as Theorem 2.
From above results immediately follows

Theorem 4. For an n-semigroup $< G, f >$, conditions (1), (2), (6) and

(7) $(\forall a \in G)\ (\exists y_2, \ldots,\ y_n \in G)\ f_{(2)}(a,\ y_2,\ a,\ y_3,\ a, \ldots,\ a,\ y_n,\ a) = a$

$$(8)\ (\forall a \in G)\ (\exists x_1, \ldots,\ x_k \in G)\begin{cases} f\,(a,\ x_1,\ a, \ldots,\ a,\ x_k,\ a) = a \text{ if } n = 2k+1 \\ f(a, x_1, x_2, a, x_3, a, \ldots, a, x_k, a) = a \text{ if } n = 2k \end{cases}$$

are equivalent. ∎

2. n-groups

An *n-group* is an n-semigroup $<G, f>$ possesing the additional property that for each $a_0,\ a_1, \ldots,\ a_{i-1},\ a_{i+1}, \ldots,\ a_n \in G$, a unique solution in the indeterminate x_i exists for the equation

$$(9) \qquad f\,(a_1^{i-1}\ x_i,\ a_{i+1}^n) = a_0,$$

for each $i = 1,\ 2, \ldots,\ n$.

It is worth while to note that it suffices only to postulate the exi-existence of a solution of (9) at the places $i=1$ and $i=n$ or at the one place i other than 1 and n. Then, one can prove uniqueness of the solution of (9) for all $i=1, 2, \dots, n$.

Theorem 5. An n-semigroup $<G, f>$ is an n-group if and only if, for some $k=1, 2, \dots, n\text{-}2$

$$(10\quad \left\{\begin{array}{l}(\forall a_1, \dots, a_k \in G)\ (\exists x_{k+1}, \dots, x_{n-1}, y_{k+1}, \dots, y_{n-1} \in G)\ (\forall b \in G)\\ f(a_1^k, x_{k+1}^{n-1}, b)=f(b, y_{k+1}^{n-1}, a_1^k)=b.\end{array}\right.$$

N. Celakoski proved in [1] a special case of this theorem. The proof in [1] is more complicated.

Proof. If $<G, f>$ is an n-group, then for all $a, a_1, \dots, a_k,\ b \in G$ there exists $x_{k+1}, \dots, x_{n-1}, z_2, \dots, z_n \in G$ such that $f\,(a_1^k, x_{k+1}^{n-1}, a)=a$ and $b=f\,(a, z_2^n)$. Hence

$$b=f(a, z_2^n)=f(f\,(a_1^k, x_{k+1}^{n-1}, a), z_2^n)=f(a_1^k, x_{k+1}^{n-1}, f(a, z_2^n))=f(a_1^k, x_{k+1}^{n-1}, b)$$

for every $b \in G$.

Similarly, we obtain the second part of (10).

On the other hand, if (10) is true, then we shall show a solution of the equation $f(z_2^n, z)=z_0$ for arbitrary $z_0, z_2, \dots, z_n \in G$.

Put

$$z=f_{(n-2)}\,(a_{n2}^{nk}, x_n^n{}_{k+1}^{n-1}, \dots, a_3^3{}_2^k, x_3^3{}_{n+1}^{n-1}, a_2^2{}_2^k, x_2^2{}_{k+1}^{n-1}, z_0)$$

where $f(z_i, a_i^i{}_2^k, x_i^i{}_{k+1}^{n-1}, b)=b$ for all $b \in G$, we get

$$f\,(z_2^n, z)=f_{(n-1)}\,(z_2^n, a_n^n{}_2^k, x_n^n{}_{k+1}^{n-1}, \dots, a_2^2{}_2^k, x_2^2{}_{k+1}^{n-1}, z_0)=$$

$$=f_{(n-2)}(z_2^{n-1}, f\,(z_n, a_n^n{}_2^k, x_n^n{}_{k+1}^{n-1}, a_{n-1\,2}), a_{n-1}^{n-1}{}_3^k, x_{n-1}^{n-1}{}_{k+1}^{n-1}, \dots, z_0)=$$

$$=\dots=f\,(z_2, a_2^2{}_n^k, x_2^2{}_{k+1}^{n-1}, z_0)=z_0.$$

In this same manner we can verify that the element

$$z'=f_{(n-2)}\,(z_0, y_n^n{}_{k+1}^{n-1}, a_n^n{}_1^{k-1}, \dots, y_3^3{}_{k+1}^{n-1}, a_3^3{}_1^{k-1}, y_2^2{}_{k+1}^{n-1}, a_2^2{}_1^{k-1})$$

where $f(b, y_i^i{}_{k+1}^{n-1}, a_i^i{}_1^{k-1}, z_i)=b$ for each $b \in G$, is a solution of the equation $f\,(z', z_2^n)=z_0$, which completes the proof. ∎

As an immediate consequence we obtain

Theorem 6. *A* regular n-semigroup is an n-group if and only if, it is cancellative.

Pfoof. The first part is trivial. To prove the second part we assume that $< G, f>$ is regular and cancellative. Then

$$f\ (b,x_3^n,a) = f\ (b, x_3^n, f(a,x_3^n, a)) = f_{(2)}(b,x_3^n, a, x_3^n, a)$$

for all $b \in G$ and some $x_3, \ldots, x_n \in G$ such that $a = f\ (a, x_3^n. a)$.

Gancellativity implies that $b = f(b, x_3^n, a)$ for all $b \in G$.

Similarly we obtain $b = f\ (a,x_3^n,b)$. ∎

Since all autodistributive and cancellative n-semiroups are regular]3], then we have

COROLLARY. An autodistributive n-semigroup is an n-group if and only if it is cancellative. ∎

3. Idemotents in reguiar n-semigroups

Elements $x_1, \ldots, x_n \in G$ are *regular conjugates* if $f(x_i^n, x_2^i) = x_i$ for each $i=2, 3, \ldots, n$. An element x_i is called regular conjugate with the sequence $x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$. An n-semigroup is regular if and only if for every element x there exists the sequence regular conjugates with x. O. W. Kolesnikov proved [6] that an n-semigroup is inverse if and only if all elements regular conjugates with given the sequence are identical.. Hence each regular and cancellative n-semigroup i inverse.

Direct computation show that

COROLLARY. If $y_2, y_3, \ldots, y_n$ and $y_2, y_3, \ldots, y_{n-1}, a,$ are regular conjugates, then $y_2, \ldots y_{n-1}, f\ (a, y_2^n)$ and $y_2, \ldots, y_{n-1}, f(y_n, y_2^{n-1}, a)$ aslo are regular conjugates. ∎

Idemotents of n-semigroups, for $n \geqslant 3$, have properties different from a binary case. For example: there exists n-groups which have several idempotents but there exists also n-groups without idempotents. An idempotent of an n-group is not necessarily an identity element. Moreover, for every n there exists an n-group (an (m,n)—ring) such that all elements are identities [3]. There exists also cyclic n-groups without idempotents [3].

It is well known that, if a and b are idempotents of a binary inverse semigroup, then ab is an idempotent and $ab=ba$. In the n-ary case, where $n \geqslant 3$, it is not true. Indeed, if $<G, f>$ is a 3-group derived from a group S_4, then

$$a=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}, \quad b=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}, \quad c=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$

are idempotents, $f(a,b,c) \neq f(c,b,a)$ and $f(a,b,c)$ is not an idempotent.

But we can prove

Theorem 7. If $b=f(b, x_3^n, b)$ for some $x_3, \ldots, x_n$, and b is an idempotent, then $f(x_i^n, b, b, x_3^{i-1})$ is an idempotent for all $3 \leqslant i \leqslant n+1$.

Conversely, if $f(x_i^n, b, b, x_3^{i-1})$ is an idempotent for some i and $b=f(b, x_3^n, b)$, where $<G, f>$ is a cancellative n-semigroup, then b is an idempotent. ∎

4. Regular (m,n)-rings

An algebraic structure $<R; g, f>$ is called an (m,n)-ring, if

(i) $<R,g>$ is a commutative m-group,

(ii) $<R,f>$ is an n-semigroup,

(iii) multiplication f is distributive with respect to g, i. e.

$$(11) \quad \begin{cases} (\forall x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in G)\ (\forall y_1, \ldots, y_m \in G)\ (\forall 1 \leqslant i \leqslant n) \\ f(x_1^{i-1}, g(y_1^m), x_{i+1}^n) = g(f(x_1^{i-1}, y_1, x_{i+1}^n), \ldots, f(x_1^{i-1}, y_m, x_{i+1}^n)). \end{cases}$$

An (m,n)-ring $<R; g, f>$ is *regular*, if $<R, f>$ is regular.

An element $x \in R$ is called an additive idempotent, if it is an idempotent in $<R,g>$.

It is easily verified that if an (m,n)-ring $<R; g, f>$ has at least one additve idempotent, then the set of these elements forms an ideal. Each ideal of this (m,n)-ring contains at least one such idempotent. If an (m,n)-ring is cancellative, then R has not additive idempotents, or R has only one such idempotent, or $<R, g>$ is an idempotent m-group [3].

Now we prove useful

Theorem 8. Every ideal of a regular commutative (m, n)-ring is a radical ideal, i. e. every ideal A is the form

$$\sqrt{A} = \{a \in R : a^{<k>} \in A. \text{ for some natural } k\}.$$

Proof. It is clear that $A \subset \sqrt{A}$. Now we prove that $\sqrt{A} \subset A$.

A simple induction show that if $a \in R \setminus A$, then we have $a^{<s>} = f_{(s)}\binom{(s(n-1)+1)}{a} \notin A$ for all natural s. Indeed, if $a^{<1>} \in A$, then regularity and commutativity implies that

$a = f_{(2)}(a, x_1, a, \ldots, a, x_n, a) = f\,(f(a, \ldots, a), x_2^n) \in A$ for some $x_2, \ldots, x_n$ what is impossible.

Assume that $a^{<s_1>}, \ldots, a^{<s_1>} \notin A$ and $s_1 \geqslant s_i$ for $i = 2, \ldots, n$. Than for some $y_2, \ldots, y_n \in R$ we have

$$a^{<s_1>} = f_{(2)}\left(a^{<s_1>}, y_2, a^{<s_1>}, \ldots, a^{<s_1>}, y_n, a^{<s_1>}\right) =$$

$$= f(f\,(a^{<s_1>}, \ldots, a^{<s_1>}), y^n)$$

$$= f\,(f\,(a^{<s_1>}, a^{<s_2>}, \ldots, a^{<s_n>}), f\,(a^{<p>}, a^{<n-2>}, a, \ldots, a, y_2), y_3^n),$$

where $p = ns_1 - (s_1 + s_2 + \ldots + s_n) - (n-2)$.

Now, if $f\,(a^{<s_1>}, a^{<s_2>}, \ldots, a^{<s_n>}) \in A$, then $a^{<s>} \in A$, too.

This contradiction completes the proof. ∎

GOROLLARY. If $a \in R$ is not an additive idempotent of a regular commutative (m, n)-ring $< R; g, f >$, then $a^{<S>}$, is not an additive idempotent for all natural S. ∎

COROLLARY. [7] A regular commutative n-semigroup with a zero has not nilpotent elements. ∎

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О ИДЕАЛАХ В РЕГУЛЯРНЫХ n-ПОЛУГРУППАХ

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Резюме

В этой работе пользуемся записью и определениями работ [4] и [2]. Мы обобшаем понятие регулярных полугрупп на n-арный случай $(n > 2)$ и доказываем, что определения (1), (2), (6), (7), (8) эквивалентны.

Мнжество $(a)_j = \{f(x_1^{j-1}, a, x_{j+1}^n) : x_k \in G\} \cup \{a\}$ называется главным j-идеалом n-полугруппы $< G, f >$ порожденным элементом $a \in G$.

Теорема 1. Для n-полугруппы $< G, f >$ следуюищие свойства эквивалентны:

(i) $< G, f >$ регулярная,

(ii) $\bigcap_{j=1}^{n} B_j \subset f(B_1, B_{n-1}, B_{n-2}, \ldots, B_2, B_n)$ для всех j-идеалов B_j,

(iii) $\bigcap_{j=1}^{n} (a_j)_j \subset f((a_1), (a_2)_{n-1}, (a_3)_{n-2}, \ldots, (a_{n-1})_2, (a_n)_n)$ для всех $a_1, a_2, \ldots, a_n \in G$,

(iv) $\bigcap_{j=1}^{n} (a)_j \subset f((a)_1\ (a)_{n-1}, (a)_{n-2}, \ldots, (a)_2, (a)_n)$ для любого $a \in G$.

Теоорема 2. Все идеалы n-полугруппы идемпотентны тогда и толко тогда, когда она регулярная.

Если в n-полугруппе $< G, f >$ уравнение (9) разрешимо однозначно для любых a_0, $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in G$ и для любого $i=1, 2, \ldots, n$, то она называется n-группой.

Методом из работ [4], [5] можно доказать, что:

Теорема 5. n-полугруппа является n-группой тогда и толко тогда, когда условие (10) исполнено для некоторого $k=1, 2, \ldots, n$-2.

Из этого вытекает что любая регулярная и сократимая n-полугруппа будет n-группой. Да е все автодистрибутивные [3] исократимые n-полугруппы являются n-группами,

Теорема Целякоского [1] является частичным случаем нашей теоремы.

Идемпотенты регулярных n-полугрупп ($n > 2$ имеют несколько отличных свойств чем в бинарном случае.

Алгебра $< R; g, f >$ называетея *регулярным* (m, n)*-кольцом,* если исполнены следующие свойства:

(i) $< R, g >$ коммутативная m-группа,

(ii) $< R, f >$ регулярная n-полугруппа,

(iii) исполнено (11).

Теорема 8. всякий идеал A регулярного коммутативного (m, n)-кольца имеет вид $\{a \in R : f_{(k)} (a, \ldots, a) \in A$ для некоторого натурального $k\}$,

Из этого следует что $f_{(k)} (a, \ldots, a)$ не будет аддитивным идемпотентом для никакого к, если $a \in R$ не будет таким идемпотентом. Регулярная коммутативная n-полугруппа с нулем не имеет нильпотенных элементов.