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SOME RESULTS ON SMOOTH MAPS*

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Abstract. In this paper, we generalize two results about harmonic maps to any smooth maps.

1. Introduction

Many mathematicians have studied harmonic maps between Riemannian manifolds (for example, see [1] and [2]). But, we have few results on smooth maps. In this paper, we generalize two results about harmonic maps to any smooth maps. In fact, we obtain the following theorems:

Theorem 1. Let (M,g) be a compact orientable Riemannian manifold and (N,h) be a complete Riemannian manifold, and $f: M \to N$ be a smooth map. If the rank of $f \le r$ $(r \ge 2)$ and there exist constants a and b such that

(i) $\mathrm{Riem}^N \le b$ $(\ge o)$,

(ii) At each point $x \in M$, $\mathrm{Ric}^M\big|_{E^+} \ge a$, where $E^+ \subsetneq T_x(M)$ is the nonzero characteristic space of f^*h, the pullback of h by f at x,

(iii) $2e(f)\left[a-2\frac{r-1}{r}be(f)\right] - \|\tau(f)\|^2 \ge 0$,

where e(f) is the energy density of f and $\tau(f)$ is the tension vector field. Then f is a constant map or a totally geodesic map of rank r.

When f is a harmonic map, Theorem 1 reduces to the following well-known result:

Corollary 1 ([1]). Let (M,g) be a compact orientable Riemannian manifold and (N,h) be a complete Riemannian manifold, and $f: M \to N$ be a harmonic map. If the rank of $f \le r$ $(r \ge 2)$ and if there exist constants a and b such that

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(i) $\mathrm{Riem}^N \leq b$ $(\geq o)$, $\mathrm{Ric}^M \geq a$,

(ii) $be(f) \leq ar/2(r-1)$,

then f is a constant map or a totally geodesic map of rank r.

Theorem 2. Let (M,g) be a connected compact orientable Riemannian manifold, (N,h) be a complete Riemannian manifold and f: $M \to N$ be a smooth map. If the rank of f $\leq r$ $(r \geq 2)$ and if there exist constants a and b such that

(i) $\mathrm{Riem}^N \leq b$ $(\geq o)$, $\mathrm{Ric}^M \geq a$,

(ii) $be(f) \leq ar/2(r-1)$, $\|B(f)\| \geq \|\tau(f)\|$,

where B(f) is the second fundamental form of f, then $be(f)=ar/2(r-1)$ and $\|B(f)\| = \|\tau(f)\|$. Moreover

(a) If there exists a point $x \in M$, $\mathrm{Ric}^M(x) > a$, then f is a constant map.

(b) If at each point $x \in N$, $\mathrm{Riem}^N < b$, then f is a constant map, of rank(f)=1.

When f is a harmonic map, we get the following result by [illegible]ting a=b=0 in Theorem 2.

Corollary 2 ([2] or [3]). Let (M,g) be a connected compact [illegible]ble Riemannian manifold, (N,h) be a complete Riemannian ma[illegible] and f: $M \to N$ be a harmonic map. Suppose that $\mathrm{Riem}^N \leq o$ and $\mathrm{Ric}^M \geq o$, then

(a[illegible] f is a totally geodesic map.

[illegible] If Ric^M is strictly positive definite at a point $x \in M$, th[illegible] constant map.

[illegible] $\mathrm{Riem}^N < o$, then f is either a constant map or of rank [illegible]ne, in which case its image is a closed geodesic.

In this paper, we also obtain the following result

[illegible]roposition 1. Let (M,g) be a connected compact orientable Ri[illegible]annian manifold, (N,h) be a complete Riemannian manifold and f: $M \to N$ be a smooth map. If

(i) $\mathrm{Riem}^N \geq o$, $\mathrm{Ric}^M \leq o$,

(ii) $\|B(f)\| \leq \|\tau(f)\|$.

Then we have $\|B(f)\| = \|\tau(f)\|$ and

(a) If there exists a point $x \in M$, $\mathrm{Ric}^M(x) < o$, then f is a constant map.

(b) If $\mathrm{Riem}^N > o$, then f is either a constant map or of rank one.

2. Basic Formulas. Let (M,g) and (N,h) be complete Riemannian manifolds with metric tensors g and h, respectively. Let $f: M \to N$ be a smooth map. Choose e_i (resp. e^*_α) to be orthonormal frame fields of M (resp. N) and ω_i (resp. ω^*_α) be the coframe fields to the frame fields e_i (resp. e^*_α). Let $f^*\omega^*_\alpha = \sum_i f_{\alpha i}\omega_i$. Exterior differentiating the formula and making use of structure equations of M and N, we can get

$$\sum_i Df_{\alpha i} \wedge \omega_i = o, \text{ where} \tag{1}$$

$$Df_{\alpha i} \equiv df_{\alpha i} + \sum_j f_{\alpha j}\omega_{ji} + \sum_\beta f_{\beta i} f^*\omega^*_{\beta\alpha} \equiv \sum_j f_{\alpha ij}\omega_j,$$

$$f_{\alpha ij} = f_{\alpha ji}. \tag{2}$$

Similiarly we can define $f_{\alpha ijk}$ as follows

$$df_{\alpha ij} + \sum_k f_{\alpha kj}\omega_{ki} + \sum_k f_{\alpha ik}\omega_{kj} + \sum_\beta f_{\beta ij} f^*\omega^*_{\beta\alpha} \equiv \sum_k f_{\alpha ijk}\omega_k.$$

We have the following Ricci formula

$$f_{\alpha ijk} - f_{\alpha ikj} = \sum_\ell f_{\alpha\ell} R^M_{\ell ijk} + \sum_{\beta,\nu,\delta} f_{\beta i} f_{\nu j} f_{\delta k} R^N_{\beta\alpha\nu\delta}, \tag{3}$$

where $1 \le i,j,k,\ 1 \le \dim(M)=n,\ 1 \le \alpha,\beta,\nu,\delta \le \dim(N)=m$.

The energy density of f is $e(f) = \frac{1}{2} \sum_{\alpha,i} f^2_{\alpha i}$, the tension vector field of f is $\tau(f) = \sum_{\alpha,i} f_{\alpha ii} e^*_\alpha$. If $\tau(f)=o$, f is called a harmonic map. Vector field $B(f) = \sum_{\alpha,i,j} f_{\alpha ij}\omega_i\omega_j e^*_\alpha$ is called the second fundamental form of f. If $B(f) \equiv o$, f is called a totally geodesic map. For any smooth map f, the Laplacian of $e(f)$ can be computed as follows

$$\begin{aligned}
\Delta e(f) &= \sum_{\alpha,i} f_{\alpha i}\Delta f_{\alpha i} + \|B(f)\|^2 = \sum_{\alpha,i,k} f_{\alpha i} f_{\alpha ikk} + \|B(f)\|^2 = \\
&= \sum_{\alpha,i} f_{\alpha i}\Big(\sum_k f_{\alpha kki} + \sum_{\ell,k} f_{\alpha\ell} R^M_{\ell kik} + \sum_{k,\beta,\nu,\delta} f_{\beta k} f_{\nu i} f_{\delta k} R^N_{\beta\alpha\nu\delta}\Big) + \\
&\quad + \|B(f)\|^2 = \\
&= \sum_{\alpha,i,k} f_{\alpha i} f_{\alpha kki} + \sum_{\alpha,i,\ell} \mathrm{Ric}^M(f_{\alpha i}e_i, f_{\alpha\ell}e_\ell) + \\
&\quad + \sum_{i,k,\beta,\alpha,\nu,\delta} \mathrm{Riem}^N(f_{\beta k}e^*_\beta, f_{\alpha i}e^*_\alpha, f_{\nu i}e^*_\nu, f_{\delta k}e^*_\delta) + \|B(f)\|^2 =
\end{aligned}$$

$$= \sum_{\alpha,i,k} f_{\alpha i} f_{\alpha kki} + \sum_i \langle f_* \mathrm{Ric}^M(e_i), f_* e_i \rangle_N - \quad (4)$$
$$- \sum_{k,i} \mathrm{Riem}^N(f_* e_k, f_* e_i, f_* e_k, f_* e_i) + \|B(f)\|^2.$$

We also need the following lemma to prove theorem 1 and theorem 2.

Lemma 1. Let (M,g) be a compact orientable Riemannian manifold, (N,h) be a complete Riemannian manifold, and $f: M \to N$ be a smooth map. Then we have

$$\int_M \sum_{\alpha,i,k} f_{\alpha i} f_{\alpha kki} *1 = -\int_M \|\tau(f)\|^2 *1. \quad (5)$$

Proof. Note that

$$\sum_{\alpha,i,k} f_{\alpha i} f_{\alpha kki} = \sum_{\alpha,i,k} (f_{\alpha i} f_{\alpha kk})_i - \|\tau(f)\|^2.$$

Integrating two sides of the above formula and using Stokes theorem (see [4]), we get (5).

3. Proof of Theorem 1. Suppose that the conditions in Theorem 1 are satisfied and f is not a constant map (i.e., $e(f) \not\equiv 0$). We prove that f must be a totally geodesic map of rank r.

Fix a point $x \in M$ and diagonalize (f^*h) at the point x. Thus we have

$$(f^*h)_x = \sum_{j=1}^{p} \lambda_j (\omega_j)^2, \quad g_x = \sum_{i=1}^{n} \omega_i^2, \quad (6)$$
$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p > 0,$$
$$(\mathrm{rank}\, df)_x = p \leq r.$$

Denote the induced map on the space of k-vectors by $(\Lambda^k df)$ ([1]) and write $|\Lambda^k df|_x^2 = \sum_{i_1 < \ldots < i_k} \lambda_{i_1} \ldots \lambda_{i_k}$. The following inequality holds

$$|\Lambda^2 df|_x^2 \leq \binom{p}{2} |df|_x^4 / p^2, \quad (7)$$

and equality holds if and only if $\lambda_1 = \ldots = \lambda_p$.

Since $p \leq r$, we have

$$|\Lambda^2 df|^2 \leq (r-1)|df|^4/2r, \quad (8)$$

where equality holds if and only if $p=r$ and $\lambda_1 = \ldots = \lambda_r$.

Thus, using (8) and conditions (i) and (ii) of Theorem 1, we can make the following computation by (4)

$$\begin{aligned}\Delta e(f) &\geq \sum_{\alpha,i,k} f_{\alpha i}f_{\alpha kki}+a|df|^2-2b|\Lambda^2 df|^2+\|B(f)\|^2 \geq \\ &\geq \sum_{\alpha,i,k} f_{\alpha i}f_{\alpha kki}+|df|^2\left(a-\frac{r-1}{r}b|df|^2\right)+\|B(f)\|^2 = \\ &= \sum_{\alpha,i,k} f_{\alpha i}f_{\alpha kki}+2e(f)\left(a-2\frac{r-1}{r}be(f)\right)+\|B(f)\|^2.\end{aligned} \tag{9}$$

Using (5) of lemma 1 and the compactness of M, we get by integrating (9):

$$0 \geq \int_M \left[\|B(f)\|^2+2e(f)\left(a-2\frac{r-1}{r}be(f)\right)-\|\tau(f)\|^2\right]*1. \tag{10}$$

Combining (10) with condition (iii) of Theorem 1, we get $B(f)=o$ and $p=r$. Thus f is a totally geodesic map of rank r. This completes the proof of Theorem 1.

4. Proof of Theorem 2. From the condition (i) of Theorem 2, we have (9) and (10). By the condition (ii) of Theorem 2, we obtain from (10)

$$\|B(f)\| = \|\tau(f)\|, \quad a = 2\frac{r-1}{r}be(f).$$

In this case, (7)-(10) are all equalities. Thus we get

$$\begin{aligned}&\sum_{\alpha,i,\ell} R^M_{i\ell}f_{\alpha i}f_{\alpha\ell}-\sum_{k,i}\mathrm{Riem}^N(f_*e_k,f_*e_i,f_*e_k,f_*e_i) = \\ &= 2e(f)\left(a-2\frac{r-1}{r}be(f)\right) = 0.\end{aligned} \tag{11}$$

If there exists a point $x\in M$, $\mathrm{Ric}^M(x) > a$, then $(df)_x=o$, otherwise we have at $x\in M$

$$\begin{aligned}&\sum_{\alpha,i,\ell} R^M_{i\ell}f_{\alpha i}f_{\alpha\ell}-\sum_{k,i}\mathrm{Riem}^N(f_*e_k,f_*e_i,f_*e_k,f_*e_i) > \\ &> 2e(f)\left(a-2\frac{r-1}{r}be(f)\right) = 0.\end{aligned} \tag{12}$$

It is a contradiction by (11). Thus we have $(df)_x=o$. By connectivity of M, we conclude that $df=o$ holds for every point on M. So f is a constant map.

If $\mathrm{Riem}^N < b$ at each point $x\in N$, then we get $\mathrm{rank}(f) \leq 1$, otherwise we have (12), which is a contradiction. Thus we get rank(f) ≤ 1. If rank(f)=o, f is a constant map, or we have rank(f)=1. This completes the proof of Theorem 2.

5. Proof od Proposition 1. By condition (i) of proposition 1, we get from (4)

$$\Delta e(f) \leq \sum_{\alpha,i,k} f_{\alpha i} f_{\alpha kki} + \|B(f)\|^2. \tag{13}$$

Using (5) of lemma 1 and the compactness of M, we obtain by integrating (13)

$$\int_M (\|B(f)\|^2 - \|\tau(f)\|^2)*1 \geq 0. \tag{14}$$

Combining (14) with condition (ii) of proposition 1, we get $\|B(f)\| = \|\tau(f)\|$. The proof of other conclusions of proposition 1 is the similar as the proof of Theorem 2, and therefore we omit it here. This completes the proof of Proposition 1.

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R E F E R E N C E S

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