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A SHIFT OPERATOR APPLICATION ON INEQUALITIES IN ℓ^2 SPACE

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Abstract. In this note some inequalities in the complex space ℓ^2 are proved. It is done by means of the shift and differential operators in ℓ^2. The results are generalisations and improvements of results from [1]. These inequalities from [1] are repeated in the Introduction.

INTRODUCTION

Theorem 1. [1]. Let $\{a_n\}$ be a sequence of real numbers such that

$$\sum_{-\infty}^{\infty} a_n^2 < \infty \text{ and } \sum_{-\infty}^{\infty} (\Delta^2 a_n)^2 < \infty.$$

Then

$$\{\sum_{-\infty}^{\infty} (\Delta a_n)^2\}^2 \leq \sum_{-\infty}^{\infty} a_n^2 \sum_{-\infty}^{\infty} (\Delta^2 a_n)^2. \tag{1}$$

Equality occurs if and only if $a_n=0$ for all $n\in N:=\{1,2,\ldots\}$.

Here and in the sequel the following notation is used:

$$\Delta a_n := a_{n+1} - a_n,\quad \Delta^k a_n := \Delta\Delta^{k-1} a_n,\quad \forall n\in Z:=\{0,\pm 1,\pm 2,\ldots\},\quad \forall k\in N.$$

Theorem 2. [1]. Let $\{a_n\}$ be a sequence of real numbers such that

$$\sum_{0}^{\infty} a_n^2 < \infty \text{ and } \sum_{0}^{\infty} (\Delta^2 a_n)^2 < \infty.$$

Then

$$\{\sum_{0}^{\infty} (\Delta a_n)^2\}^2 \leq 4\sum_{0}^{\infty} a_n^2 \sum_{0}^{\infty} (\Delta^2 a_n)^2. \tag{2}$$

Equality occurs if and only if $a_n=0$ for all $n\in N\cup\{0\}$.

The inequalities (1) and (2) are analogous to integral inequalities from [2].

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THE PROOFS AND FORMULATIONS OF THE MAIN STATEMENTS

1. Let $(\ell^2,(.,.))$ denote the Hilbert space of complex sequences. Let us remind the reader that $x:=\{x_n\}_{-\infty}^{\infty}$ belongs to ℓ^2 if and only if $\sum_{n\in Z} |x_n|^2 < \infty$. The Scalar product is defined by

$$(x,y) := \sum_{n\in Z} x_n \overline{y}_n \quad (x,y\in\ell^2).$$

Let e^n denote a sequence which satisfies $e_j^n = \delta_j^n$ $(n,j\in Z)$, where δ_j^n is Kronecker symbol. Obviously $\{e^n\in\ell^2 \mid n\in Z\}$ is a topological base of ℓ^2. Let us introduce the operator $V: \ell^2 \to \ell^2$ by

$$V(\sum_{n\in Z} x_n e^n) := \sum_{n\in Z} x_{n-1} e^n.$$

The adjoint operator of V is denoted by V^*. This means $(Vx,y)=(x,V^*y)$ holds for all $x,y\in\ell^2$. This implies $V^*(\sum_{n\in Z} x_n e^n)= \sum_{n\in Z} x_{n+1} e^n$. Hence, $V^*V = VV^* = I$, where I denotes the identity operator. This means that V and V^{*k} are unitary operators. All these definitions and statements can be seen in many books of functional analysis, for example in [3]. In order to prove the Theorem 1.2 below we need to introduce the operator $\Delta:=V^*-I$. Obviously $\Delta x= \sum_{n\in Z} \Delta x_n e^n$. It is not difficult to prove: $\Delta(\ell^2)\subseteq\ell^2$ and $\Delta^{-1}(\ell^2)\neq\ell^2$. Indeed,

$$x\in\ell^2 \Rightarrow \sum_{n\in Z} |x_n|^2 < \infty \Rightarrow \sum_{n\in Z} |x_{n+1}-x_n|^2 \leq \sum_{n\in Z} |x_{n+1}|^2 + \sum_{n\in Z} |x_n|^2 < \infty \Rightarrow$$

$$\Rightarrow \Delta x\in\ell^2.$$

The converse implication: $\Delta x\in\ell^2 \Rightarrow x\in\ell^2$ does not hold. Indeed, for the sequence

$$x = (\ldots,0,1,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{3}},\ldots,\frac{1}{\sqrt{n}},\ldots)$$

it holds $\Delta x\in\ell^2$ and $x\notin\ell^2$.

Obviously, it holds: $x\in\ell^2 \Rightarrow \Delta^k x\in\ell^2$, $\forall k\in N$.

We shall need the following lemma.

Lemma 1.1. a) For the point spectrum of Δ and Δ^* it holds

$$\delta_p(\Delta) = \delta_p(\Delta^*) = \emptyset.$$

b) $\quad 0\notin\delta_p(\Delta^k), \quad 0\notin\delta_p(\Delta^{*k}), \ \forall k\in N.$

<u>Proof</u>. a) $\lambda\in\delta_p(\Delta) \Leftrightarrow (\exists x\in\ell^2)(\Delta x=\lambda x) \Leftrightarrow x_{n+1}-x_n = \lambda x_n,\ \forall n\in Z$ $\Leftrightarrow x_{n+1}=(1+\lambda)x_n,\ \forall n\in Z$. Obviously, $x_n = (1+\lambda)^n x_0,\ \forall n\in Z$. This implies

$$\sum_{n\in Z}|x_n|^2 = \sum_{n=0}|1+\lambda|^{2n}|x_0|^2 + \sum_{n=-1}|1+\lambda|^{2n}|x_0|^2 =$$

$$= |x_0|^2\sum_{n=0}^{\infty}|1+\lambda|^{2n}+|x_0|^2\sum_{n=1}^{\infty}|1+\lambda|^{-2n} = \infty,\ \forall\lambda\in C.$$

Hence, $x\notin\ell^2$, i.e. $\delta_p(\Delta) = \emptyset$. By the same arguments $\delta_p(\Delta^*)$ is proved.

b) $0\in\delta_p(\Delta^k) \Rightarrow (\exists x\in\ell^2)\Delta^k x = 0 \Rightarrow \exists m\in\{1,2,\dots,k\}$ such that $\Delta^m x = 0$ and $\Delta^{m-1}x \neq 0$. This means that $\Delta^{m-1}x$ is an eigenvector of the operator Δ in $\lambda=0$, which is according to a) impossible. Hence, $0\notin\delta_p(\Delta^k)$. On the same way one proves $0\notin\delta_p(\Delta^{*k}),\ \forall k\in N$.

<u>Theorem 1.2</u>. Let $\{x_n\}_{-\infty}^{\infty}$ and $\{y_n\}_{-\infty}^{\infty}$ be sequences from ℓ^2. Then the following inequalities hold

a) $|\sum_{n\in Z}\Delta^k x_n\Delta^{\bar{k}} y_n|^2 \le (\sum_{n\in Z}|x^n|^2)(\sum_{n\in Z}|\Delta^{2k}y_n|^2)\quad (k\in N)$.

The equality holds if and only if x=0 or y=0 or $x=\lambda y$ for some $\lambda\in C$.

b) $\sqrt{\sum_{n\in Z}|\Delta^k(x_n+y_n)|^2} \le \sqrt{\sum_{n\in Z}|\Delta^k x_n|^2} + \sqrt{\sum_{n\in Z}|\Delta^k y_n|^2}\quad (k\in N)$.

The equality holds if and only if x=0 or y=0 or $x=\lambda y$ for some $\lambda\in C$.

<u>Proof</u>.

a) $$|\sum_{n\in Z}(\Delta^k x_n)(\Delta^k\bar{y}_n)|^2 = |(\Delta^k x,\Delta^k y)|^2 = |(x,\Delta^{k*}\Delta^k y)|^2 \le$$
$$\le (x,x)(\Delta^{k*}\Delta^k y,\Delta^{k*}\Delta^k y)$$

Besides, as V^{k*} is unitary operator we have

$$(\Delta^{k*}\Delta^k y,\Delta^{k*}\Delta^k y) = (V^{*k}\Delta^{*k}\Delta^k y,V^{*k}\Delta^{*k}\Delta^k y) =$$
$$= (V^{*k}(V-I)^k(V^*-I)^k y,V^{*k}(V-I)^k(V^*-I)^k y) =$$
$$= ((I-V^*)^k(V^*-I)^k y,(I-V^*)^k(V^*-I)^k y) =$$
$$= (\Delta^{2k}y,\Delta^{2k}y).$$

Hence

$$|\sum_{n\in Z}(\Delta^k x_n)(\Delta^k\bar{y}_n)|^2 \le (x,x)(\Delta^{2k}y,\Delta^{2k}y) =$$
$$= (\sum_{n\in Z}|x_n|^2)(\sum_{n\in Z}|\Delta^{2k}y_n|^2).$$

According to a well known property of Cauchy inequality, the equality in the first line of the proof holds if and only if $\Delta^k x=0$ or $\Delta^k y=0$ or $\Delta^k x=\lambda\Delta^k y$ for some $\lambda\in C$. As (according to Lemma 1.1) $0\notin\delta_p(\Delta^k)$ this is equivalent to $x=0$ or $y=0$ or $x=\lambda y$ for the $\lambda\in C$.

b) This inequality follows from the Minskowsky's inequality $\sqrt{(\Delta^k(x+y),\Delta^k(x+y))} \le \sqrt{(\Delta^k x,\Delta^k x)}+\sqrt{(\Delta^k y,\Delta^k y)}$ and Lemma 1.1.

Remarks 1.3. The inequality (1) follows from a) for k=1 and $x_n=y_n=a_n\in R$, $\forall n\in Z$, which means that our Theorem 1.2 is improvement of Theorem 1 [1]. As we did not assume $\sum_{n\in Z}(\Delta^2 a_n)^2<\infty$ it is also a generalisation.

2. Let us now consider the Hilbert space $(\ell^2,(.,.))$, where

$$\ell^2:=\{\{x_n\}_{n=0}^{\infty}\in C^N:\ \sum_{n=0}^{\infty}|x_n|^2<\infty\}.$$

The elements of ℓ^2 will be denoted by letters x,y,... . The scalar product (.,.) is defined by the relation

$$(x,y):=\sum_{n=0}^{\infty}x_n\bar{y}_n.$$

Let S denote the unilateral shift, i.e. $S((x_0,x_1,\dots)):=(0,x_0,x_1,\dots)$. Then for the adjoint operators S^* it holds $S^*((x_0,x_1,\dots))=(x_1,x_2,\dots)$, $S^*S=I$ and $SS^*=P$, where $P((x_0,x_1,\dots))=(0,x_1,x_2,\dots)$. One easily verifies $P^2=P$ and $P^*=P$, i.e. P is a projection. For the operator $\Delta:=S^*-I$ it hold:

$$\Delta((x_0,x_1,\dots))=(x_1-x_0,x_2-x_1,\dots)=(\Delta x_0,\Delta x_1,\dots).$$

Remark 2.1. As before we have

$$\sum_{n=0}^{\infty}|x_n|^2<\infty\Rightarrow\sum_{n=0}^{\infty}|\Delta^k x_n|^2<\infty,\ \forall k\in N.$$

From this implication for k=2, it follows that Theorem 2. [1] remains val: if the condition $\sum_{n=0}^{\infty}(\Delta^2 a_n)^2<\infty$ is omitted.

Remark 2.2. The constant on the right hand side (number 4) in the inequality (2) can not be improved in general (see [1]). In the sequel we shall see that for the sequences $\{a_n\}_{n=0}^{\infty}\in\ell^2$ which satisfy $a_1-a_0=0$ the number 4 can be replaced with 1.

Lemma 2.3. Under the notation of the section 2 it holds:

a) $\delta_p(\Delta) = \{\lambda\in C\colon |1+\lambda| < 1\}$,

b) $\delta_p(\Delta^*) = \emptyset$.

Proof. a) $\lambda\in\delta_p(\Delta)$ if and only if $\exists x\in\ell^2$, $x\neq 0$, such that $\Delta x=\lambda x$, i.e. $x_{n+1}-x_n=\lambda x_n$, $\forall n\in\{0,1,2,\ldots\}$. This means $x_n=(1+\lambda)^n x_0$, $\forall n\in\{0,1,\ldots\}$ i.e. $\sum_{n=0}^{\infty}|x_n|^2 = |x_0|^2\sum_{n=0}^{\infty}|1+\lambda|^{2n}$. As the last series converges if and only if $|1+\lambda| < 1$, the first statement is proved.

b) $\lambda\in\delta_p(\Delta^*)$ if and only if it exists $x\in\ell^2$, $x\neq 0$, such that $\Delta^* x=\lambda x$, i.e. $x_{k-1}=(1+\lambda)x_k$, $k=0,1,2,\ldots,x_{-1}=0$. Suppose x_m is the first nonzero element in the sequence x. Then $\lambda=-1$ follows for $k=m$ and $x_m=0$ follows for $k=m+1$. Hence, $x=0$. This proves b).

Theorem 2.4. Let $x=\{x_n\}_0^\infty$ and $y=\{y_n\}_0^\infty$ be sequences in ℓ^2. Then it holds

$$\Big|\sum_{n=0}^{\infty}\Delta x_n\Delta\bar{y}_n\Big|^2 \le \Big(\sum_{n=0}^{\infty}|x_n|^2\Big)\Big(\sum_{n=0}^{\infty}|\Delta^2 y_n|^2+|\Delta y_0|^2\Big). \quad (3)$$

The equality holds if and only if $x=0$ or $y=0$ or $\lambda x=\Delta^*\Delta y$, for some $\lambda\in C$.

Proof. $(I-S^*)(P-S) = I-S^*+P-S \quad (4)$

$$\Big|\sum_{n=0}^{\infty}\Delta x_n\Delta\bar{y}_n\Big|^2 = |(\Delta x,\Delta y)|^2 = |(x,\Delta^*\Delta y)|^2$$

According to Cauchy inequality we have

$$|(\Delta x,\Delta y)|^2\le(x,x)(\Delta^*\Delta y,\Delta^*\Delta y)=(x,x)((S-I)(S^*-I)y,(S-I)(S^*-I)y)=$$
$$=(x,x)((P-S^*-S+I)y,(P-S^*-S+I)y)\overset{(4)}{=}(x,x)((I-S^*)(P-S)y,(I-S^*)(P-S)y)=$$
$$=(x,x)(S(I-S^*)(P-S)y,S(I-S^*)(P-S)y)=(x,x)((S-P)(P-S)y,(S-P)(P-S)y)=$$
$$=(x,x)((S-P)(0,y_0-y_1,y_1-y_2,\ldots),(S-P)(0,y_0-y_1,y_1-y_2,\ldots))=$$
$$=(x,x)((0,y_1-y_0,y_2-2y_1+y_0,y_3-2y_2+y_1,\ldots),(0,y_1-y_0,y_2-2y_1+y_0,y_3-2y_2+y_1,\ldots))=$$
$$=(x,x)\Big\{|y_1-y_0|^2+\sum_{n=0}^{\infty}|\Delta^2 y_n|^2\Big\}.$$

Hence, $|(\Delta x,\Delta y)|^2 \le (x,x)\{|\Delta y_0|^2 + (\Delta^2 y,\Delta^2 y)\}$, which proves (3).

According to a well known property of Cauchy inequality the equality holds if and only if $x=0$ or $\Delta^*\Delta y=0$ or $\lambda x=\Delta^*\Delta y$, for some $\lambda\in C$. According to Lemma 2.3 b) $\Delta^*\Delta y=0$ is equivalent to $\Delta y=0$. According to Lemma 2.3 a) $0\notin\delta_p(\Delta)$. Hence, $\Delta y=0$ if and only if $y=0$. This proves the last statement in theorem.

Corollary 2.5. If $a=\{a_n\}_o^\infty \in \ell^2$ and $a_o=a_1$ then it holds

$$\sum_{n=o}^{\infty} |\Delta a_n|^2 \le \left(\sum_{n=o}^{\infty} |a_n|^2\right) \sum_{n=o}^{\infty} |\Delta^2 a_n|^2. \qquad (5)$$

The equality holds if and only if $a_n=0$ for all $n=2,3,\ldots$.

Proof. For $x=y=a$ the relation (5) follows from (3) directly. The equality holds if and only if $a=0$ or $\lambda a=\Delta^*\Delta a$, for some $\lambda \in C$. But $\lambda a=\Delta^*\Delta a$ is equivalent to $a_1+(\lambda-1)a_o=0$ and $a_{k+1}+(\lambda-2)a_k+a_{k-1}=0$, $k=1,2,\ldots$. As $a_1=a_o$ we conclude $\lambda=1$ and $0=a_2=a_3=\ldots$. Obviously, this contains the case $a=0$, which proves the last statement of the corollary.

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ПРИМЕНА НА ШИФТ ОПЕРАТОРИ НАД НЕРАВЕНСТВА ВО ℓ^2 ПРОСТОРОТ

М. Бороговац

Р е з и м е

Во овој труд се докажани неколку неравенства во комплексниот простор ℓ^2. Тие се добиени со користење на шифт оператори и диференцијални оператори во ℓ^2. Овие резултати претставуваат генерализација и подобрувања на резултатите од [1], коишто се набројуваат во воведот на трудов.

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