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IMPROVEMENTS OF HÖLDER'S AND MINKOWSKI'S INEQUALITIES

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Abstract. Improvements of the well-known Hölder and Minkowski inequalities are obtained.

1. Introduction. S.S.Dragomir and Š.Z.Arslanagić [1] proved the following results:

Let $a,b\in R^n$ and $u,v\in R_+^n$ with $u \geq v$, i.e. $u_i \geq v_i$ $(i=1,\dots,n)$. Then

$$\begin{aligned}(\sum_{i=1}^n u_i a_i^2)^{1/2}(\sum_{i=1}^n u_i b_i^2)^{1/2}-|\sum_{i=1}^n u_i a_i b_i| &\geq \\ \geq (\sum_{i=1}^n v_i a_i^2)^{1/2}(\sum_{i=1}^n v_i b_i^2)^{1/2}-|\sum_{i=1}^n v_i a_i b_i| &\geq 0\end{aligned} \tag{1}$$

This result is an improvement of Cauchy-Buniakowski-Schwarz's inequality. As a consequence of this result Dragomir and Arslanagić obtained:

Let a,b,u,v be as in previous result. Then

$$\begin{aligned}((\sum_{i=1}^n u_i a_i^2)^{1/2}+(\sum_{i=1}^n u_i b_i^2)^{1/2})^2-\sum_{i=1}^n u_i(a_i+b_i)^2 &\geq \\ \geq ((\sum_{i=1}^n v_i a_i^2)^{1/2}+(\sum_{i=1}^n v_i b_i^2)^{1/2})^2-\sum_{i=1}^n v_i(a_i+b_i)^2 &\geq 0\end{aligned} \tag{2}$$

In this paper we shall give related improvements of Hölder's and Minkowski's inequalities.

2. Main results

Theorem 1. Let $a,b,u,v\in R_+^n$ with $u \geq v$, i.e. $u_i \geq v_i$ $(i=1,\dots,n)$ and $p,q\in R$ with $\frac{1}{p}+\frac{1}{q}=1$. If $p > 1$, then

$$\begin{aligned}(\sum_{i=1}^n u_i a_i^p)^{1/p}(\sum_{i=1}^n u_i b_i^q)^{1/q}-\sum_{i=1}^n u_i a_i b_i &\geq \\ \geq (\sum_{i=1}^n v_i a_i^p)^{1/p}(\sum_{i=1}^n v_i b_i^q)^{1/q}-\sum_{i=1}^n v_i a_i b_i &\geq 0\end{aligned} \tag{3}$$

If $p < 1$ $(\neq 0)$ then the reverse inequalities are valid.

Proof. We have

$$\sum_{i=1}^{n} u_i a_i b_i - \sum_{i=1}^{n} v_i a_i b_i = \sum_{i=1}^{n} (u_i - v_i) a_i b_i \le$$

$$\le \left(\sum_{i=1}^{n} (u_i - v_i) a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} (u_i - v_i) b_i^q\right)^{1/q} = \tag{4}$$

$$= \left(\sum_{i=1}^{n} u_i a_i^p - \sum_{i=1}^{n} v_i a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} u_i b_i^q - \sum_{i=1}^{n} v_i b_i^q\right)^{1/q},$$

where we used the same Hölder inequality.

Note that the following result is a special case of the Popoviciu generalization of Aczel's inequality ([2], see also [3]):

Let $A > B > 0$, $C > D > 0$, $\frac{1}{p} + \frac{1}{q} = 1$. If $p > 1$ then

$$(A^p - B^p)^{1/p} (C^q - D^q)^{1/q} \le AC - BD \tag{5}$$

If $p < 1$ ($\neq 0$) then the reverse inequality in (5) is valid.

Now, by substitutions:

$$A = \left(\sum_{i=1}^{n} u_i a_i^p\right)^{1/p}, \quad B = \left(\sum_{i=1}^{n} v_i a_i^p\right)^{1/p},$$

$$C = \left(\sum_{i=1}^{n} u_i b_i^q\right)^{1/q}, \quad D = \left(\sum_{i=1}^{n} v_i b_i^q\right)^{1/q}$$

we get

$$\left(\sum_{i=1}^{n} u_i a_i^p - \sum_{i=1}^{n} v_i a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} u_i b_i^q - \sum_{i=1}^{n} v_i b_i^q\right)^{1/q} \le$$

$$\le \left(\sum_{i=1}^{n} u_i a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} u_i b_i^q\right)^{1/q} - \left(\sum_{i=1}^{n} v_i a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} v_i b_i^q\right)^{1/q} \tag{6}$$

Now, (4) and (6) give the first inequality in (3). The second inequality in (3) is the same Hölder inequality.

Moreover, if we in the previous proof use Hölder's inequality for several sequences and analogous generalization of (5) from [4] we can obtain the following:

Theorem 2. Let $a_1, \ldots, a_m, u, v \in R_+^n$ with $u \ge v$ and let positive numbers $p_1, \ldots, p_m$ satisfy $p_1^{-1} + \ldots + p_m^{-1} = 1$. Then

$$\prod_{j=1}^{m}\left(\sum_{i=1}^{n}u_i a_{ji}^{p_j}\right)^{1/p_j}-\sum_{i=1}^{n}u_i\left(\prod_{j=1}^{m}a_{ji}\right)\geq$$

$$\geq\prod_{j=1}^{m}\left(\sum_{i=1}^{n}v_i a_{ji}^{p_j}\right)^{1/p_j}-\sum_{i=1}^{n}v_i\left(\prod_{j=1}^{m}a_{ji}\right)\geq 0 \tag{7}$$

<u>Theorem 3</u>. If in Theorem 2 $a_1,\ldots,a_m$ are complex sequences, then

$$\prod_{j=1}^{m}\left(\sum_{i=1}^{n}u_i|a_{ji}|^{p_j}\right)^{1/p_j}-\left|\sum_{i=1}^{n}u_i\left(\prod_{j=1}^{m}a_{ji}\right)\right|\geq$$

$$\geq\prod_{j=1}^{m}\left(\sum_{i=1}^{n}v_i|a_{ji}|^{p_j}\right)^{1/p_j}-\left|\sum_{i=1}^{n}v_i\left(\prod_{j=1}^{m}a_{ji}\right)\right|\geq 0 \tag{8}$$

<u>Proof</u>. We have

$$\left|\sum_{i=1}^{n}u_i\left(\prod_{j=1}^{m}a_{ji}\right)\right|-\left|\sum_{i=1}^{n}v_i\left(\prod_{j=1}^{m}a_{ji}\right)\right|\leq$$

$$\leq\left|\sum_{i=1}^{n}u_i\left(\prod_{j=1}^{m}a_{ji}\right)-\sum_{i=1}^{n}v_i\left(\prod_{j=1}^{m}a_{ji}\right)\right|= \qquad \text{(by triangle inequality)}$$

$$=\left|\sum_{i=1}^{n}(u_i-v_i)\left(\prod_{j=1}^{m}a_{ji}\right)\right|\leq$$

$$\leq\sum_{i=1}^{n}(u_i-v_i)\left(\prod_{j=1}^{m}|a_{ji}|\right)= \qquad \text{(by triangle inequality)}$$

$$=\sum_{i=1}^{n}u_i\left(\prod_{j=1}^{m}|a_{ji}|\right)-\sum_{i=1}^{n}v_i\left(\prod_{j=1}^{m}|a_{ji}|\right)\leq$$

$$\leq\prod_{j=1}^{m}\left(\sum_{i=1}^{n}u_i|a_{ji}|^{p_j}\right)^{1/p_j}-\prod_{j=1}^{m}\left(\sum_{i=1}^{n}v_i|a_{ji}|^{p_j}\right)^{1/p_j} \qquad \text{(by (7))}$$

<u>Remark</u>. Theorem 3 is an obvious generalization of (1).

<u>Theorem 4</u>. Let $a,b,u,v\in R_+^n$ with $u\geq v$. If $p\geq 1$ (or $p<0$), then

$$\left(\left(\sum_{i=1}^{n}u_i a_i^p\right)^{1/p}+\left(\sum_{i=1}^{n}u_i b_i^p\right)^{1/p}\right)^p-\sum_{i=1}^{n}u_i(a_i+b_i)^p\geq$$

$$\geq\left(\left(\sum_{i=1}^{n}v_i a_i^p\right)^{1/p}+\left(\sum_{i=1}^{n}v_i b_i^p\right)^{1/p}\right)^p-\sum_{i=1}^{n}v_i(a_i+b_i)^p\geq 0 \tag{9}$$

If $0<p<1$ we have the reverse inequalities in (9).

Proof. If $p \geq 1$ (or $p < 0$) we have

$$\sum_{i=1}^{n} u_i(a_i+b_i)^p - \sum_{i=1}^{n} v_i(a_i+b_i)^p = \sum_{i=1}^{n} (u_i-v_i)(a_i+b_i)^p \leq$$

$$\leq ((\sum_{i=1}^{n} (u_i-v_i)a_i^p)^{1/p} + (\sum_{i=1}^{n} (u_i-v_i)b_i^p)^{1/p})^p = \tag{10}$$

$$= ((\sum_{i=1}^{n} u_i a_i^p - \sum_{i=1}^{n} v_i a_i^p)^{1/p} + (\sum_{i=1}^{n} u_i b_i^p - \sum_{i=1}^{n} v_i b_i^p)^{1/p})^p$$

where we used the same Minkowski's inequality.

Note that the following result is a special case of the Bellman inequality ([5], see also [3]):

Let $A > B > 0$, $C > D > 0$. If $p \geq 1$ (or $p < 0$), then

$$((A^p-B^p)^{1/p} + (C^p-D^p)^{1/p})^p \leq (A+C)^p - (B+D)^p \tag{11}$$

If $0 < p < 1$, then the reverse inequality in (11) is valid.

Now, by substitutions:

$$A = (\sum_{i=1}^{n} u_i a_i^p)^{1/p}, \qquad B = (\sum_{i=1}^{n} v_i a_i^p)^{1/p},$$

$$C = (\sum_{i=1}^{n} u_i b_i^p)^{1/p}, \qquad D = (\sum_{i=1}^{n} v_i b_i^p)^{1/p}$$

we get from (11):

$$((\sum_{i=1}^{n} u_i a_i^p - \sum_{i=1}^{n} v_i a_i^p)^{1/p} + (\sum_{i=1}^{n} u_i b_i^p - \sum_{i=1}^{n} v_i b_i^p)^{1/p})^p \leq$$

$$\leq ((\sum_{i=1}^{n} u_i a_i^p)^{1/p} + (\sum_{i=1}^{n} u_i b_i^p)^{1/p})^p - ((\sum_{i=1}^{n} v_i a_i^p)^{1/p} + (\sum_{i=1}^{n} v_i b_i^p)^{1/p})^p \tag{12}$$

Now, (10) and (12) give the first inequality in (9). The second inequality in (9) is the same Minkowski inequality. For $0 < p < 1$ we have the reverse inequalities in (10) and (12) and so in (9).

Theorem 5. Let $a,b \in C^n$, $u,v \in R_+^n$ with $u \geq v$ and $p \geq 1$, then

$$\left(\left(\sum_{i=1}^{n} u_i|a_i|^p\right)^{1/p}+\left(\sum_{i=1}^{n} u_i|b_i|^p\right)^{1/p}\right)^p-\sum_{i=1}^{n} u_i|a_i+b_i|^p \geq$$

$$\geq \left(\left(\sum_{i=1}^{n} v_i|a_i|^p\right)^{1/p}+\left(\sum_{i=1}^{n} v_i|b_i|^p\right)^{1/p}\right)^p-\sum_{i=1}^{n} v_i|a_i+b_i|^p \geq 0 \tag{13}$$

Proof. We have

$$\sum_{i=1}^{n} u_i|a_i+b_i|^p-\sum_{i=1}^{n} v_i|a_i+b_i|^p=\sum_{i=1}^{n} (u_i-v_i)|a_i+b_i|^p \leq$$

$$\leq \sum_{i=1}^{n} (u_i-v_i)(|a_i|+|b_i|)^p = \qquad \text{(by triangle inequality)}$$

$$= \sum_{i=1}^{n} u_i(|a_i|+|b_i|)^p-\sum_{i=1}^{n} v_i(|a_i|+|b_i|)^p \leq$$

$$\leq \left(\left(\sum_{i=1}^{n} u_i|a_i|^p\right)^{1/p}+\left(\sum_{i=1}^{n} u_i|b_i|^p\right)^{1/p}\right)^p -$$

$$- \left(\left(\sum_{i=1}^{n} v_i|a_i|^p\right)^{1/p}+\left(\sum_{i=1}^{n} v_i|b_i|^p\right)^{1/p}\right)^p \qquad \text{(by (9))}$$

Remark. (13) is an obvious generalization of (2).

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ПОДОБРУВАЊА НА НЕРАВЕНСТВАТА НА HÖLDER И MINKOWSKI

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Резиме

Во оваа работа дадени се подобрувања на познатите неравенства на Hölder и Minkowski.

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