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POLYNOMIAL SOLUTIONS OF ALGEBRAIC ORDINARY DIFFERENTIAL EQUATIONS. II

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Abstract. Bordered method of definition of degrees of polynomial solutions of algebraic differential equations and entire solutions of irreducible Painleve's equations are considered. The coefficients of highest term of polynomial solutions and the quantity of polynomial solutions of different degrees of algebraic differential equations are found.

§2. BORDERED METHOD OF DEFINITION OF DEGREES OF POLYNOMIAL SOLUTIONS

By the sources of bordered method of the definition of degrees of polynomial solutions there is a method with the help of which the author and its pupils proved the theorem about the absence of entire transcendent solutions of algebraic differential equation [59,63] and so generalized known G.Wittich's theorem [82] and later spreaded this method for the establishing of the borders in which the growthis characteristics (an order and a type) of entire transcendent solutions both algebraic and non-algebraic differential equations are included [77,79].

These are following theorems in algebraic case

Theorem 2.1. Let following correlations are fulfiled:

1) $x_o=\ldots=x_p = d,\ x_\varepsilon < d,\ 0 \le p \le N,\ \varepsilon = \overline{p+1,N}$;

2) $m_o=\ldots=m_h = m,\ m_j < m,\ 0 \le h \le p,\ j = \overline{h+1,p}$;

3) $b_o=\ldots=b_\lambda = b,\ b_\tau < b,\ 0 \le \lambda \le h,\ \tau = \overline{\lambda+1,h}$;

4) $b_j - m_j \le b - m,\quad j = \overline{h+1,p}$;

5) $\sum\limits_{\ell=o}^{\lambda} \beta_\ell \ne 0$.

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Then algebraic differential equation (1) have not entire transcendent solutions.

From theorem 2.1 for $p = 1$ it's follows

G.Wittich's theorem. Algebraic differential equation (1) with one dominant term have not entire transcendent solutions.

Theorem 2.2. Let following correlations are fulfiled:

1) $x_o = \ldots = x_p = d$, $d > x_\varepsilon$, $0 < p \le N$, $\varepsilon = \overline{p+1, N}$;
2) $m_o = \ldots = m_h = m$, $m > m_j$, $0 \le h < p$, $j = \overline{h+1, p}$;
3) $b_o = \ldots = b_\lambda = b$, $b > b_\tau$, $0 \le \lambda < h$, $\tau = \overline{\lambda+1, h}$;
4) $\sum_{\ell=o}^{\lambda} \beta_\ell \ne 0$;
5) there is $j \in \{h+1, \ldots, p\}$ such that $b - m < b_j - m_j$.

Then algebraic differential equation (1) can have entire transcendent solutions of orders

$$\Omega \le \max\{M_{oj} : j = \overline{h+1, p}\}.$$

Theorem 2.3. Let following correlations are fulfiled:

1) $x_o = \ldots = x_p = d$, $d > x_\varepsilon$, $0 < p \le N$, $\varepsilon = \overline{p+1, N}$;
2) $m_o = \ldots = m_h = m$, $m < m_j$, $0 \le h < p$, $j = \overline{h+1, p}$;
3) $b_o = \ldots = b_\lambda = b$, $b > b_\tau$, $0 \le \lambda \le h$, $\tau = \overline{\lambda+1, h}$;
4) $\sum_{\ell=o}^{\lambda} \beta_\ell \ne 0$.

Then algebraic differential equation (1) can have entire transcendent solutions of orders

$$\Omega \ge \min\{M_{oj} : j = \overline{h+1, p}\}.$$

These theorems 2.1-2.3 are proved by indirect method with using of asymptotic formula of derivative's representation of entire transcendent function by means of this function itself and its Viman-Valiron's central index [82,84].

Completely analogously theorems 2.4 and 2.5 forming the essence of bordered method of definition of degrees of polynomial solutions asymptotic are proved by indirect method and in the base of proofs there is formula of derivative's representation of polynomial's derivative by means of this polynomial itself

$$w^{(n)}(z) = (-1)^n(-m)_n z^{-n} w(z) \{1 + \varepsilon_n(z)\}, \tag{2.1}$$

where $\varepsilon_o(z)=0$, for $n \ge 1$ for $z \to \infty$ rational function $\varepsilon_n(z) \to 0$; $(a)_n$ is Pochhammer's symbol.

Substituting polynomial $w(z)$ of degree deg $w=m$ in equation (1) taking into consideration (2.1) we get some identity on z. Then from this identity by indirect method we get all those statements, which are formulated in theorems 2.4 and 2.5 [63,68].

Theorem 2.4. Let following conditions are fulfiled:

1) $x_\tau > x_p = x_{p+1} = \ldots = x_{p+s} = d_p > x_\eta$, $\tau = \overline{0,p-1}$, $0 \le p \le N$, $0 \le s \le N-p$, $\eta = \overline{p+s+1,N}$;
2) $b_p - m_p > b_{p+h} - m_{p+h}$, $h = \overline{1,s}$.

Then following statements are right:

a) for $p = 0$, $s = N$ equation (1) can have polynomial solutions (3) of degrees $m < L_o$;
b) for $p = 0$, $0 \le s \le N$ the number $m \ge L$;
c) for $0 < p \le N$, $p + s = N$ the number $m \ge L$;
d) for $0 < p < N$, $p + S < N$ the number $m \ge L$, such that $m < \mathcal{U}_p$;
e) for $0 < p < N$, $p + s < N$ the number $m \ge L$, such that $\mathcal{B}_{ps}$, can be the degree of polynomial solution (3) of equation (1) if, accordingly,

b) $m \le \mathcal{B}_{os}$; c) $m \ge \mathcal{U}_p$; d) $m \le \mathcal{B}_{ps}$; e) $m \ge \mathcal{U}_p$.

Theorem 2.5. Let following conditions are fulfiled:

1) $x_\tau > x_p = x_{p+1} = \ldots = x_{p+s} = d_p > x_\eta$, $\tau = \overline{0,p-1}$, $0 \le p < N$, $0 < s \le N-p$, $\eta = \overline{p+s+1,N}$;
2) $b_p - m_p = b_{p+1} - m_{p+1} = \ldots = b_{p+\lambda} - m_{p+\lambda} > b_{p+j} - m_{p+j}$, $1 \le \lambda \le s$, $j = \overline{\lambda+1,s}$.

Then following statements are right:

a) for $p = 0$, $s = N$ the number $m \ge L$;
b) for $p = 0$, $0 < s < N$ the number $m \ge L$ and $m > \mathcal{B}_{os}$;
c) for $0 < p < N$, $p + s = N$ the number $m \ge L$ and $m < \mathcal{U}_p$;
d) for $0 < p < N$, $p + s < N$ the number $m \ge L$, $m < \mathcal{U}_p$, $m > \mathcal{B}_{ps}$, can be the degree of polynomial solutions (3) of equation (1), if it is the root of equation

$$\sum_{\xi=o}^{\lambda} \tilde{K}_{p+\xi}(m) = 0.$$

The peculiarity of bordered method formulated in theorems 2.4 and 2.5 as distinct from giving in §1, is that only one term (with number p) or the block of terms (with numbers p,p+1,...,p+s), but not all the terms of equation (1) at the same time are took in to the base of reasonings. The borders in which the degrees of polynomial solutions (3) change, depend on that in which correlation there is considering term (block of terms). Consequantly looking over all terms of equation (1) every time we define more precisely or confirm the borders of changing of the degrees of polynomial solutions.

From theorem 2.4 two statements that it is conveniently used, saying, in case of the equations of Riccati-Abel's type, follow.

Consequence 2.1. If equation (1) contain only one dominant term with number k, then the degrees $m \geq L$ of its polynomial solutions (3) satisfy to inequality

$$m \leq \max\{M_{ki}\colon i = \overline{0,N},\ k \neq i\}.$$

Consequence 2.2. If equation (1) contain only one minorant term with number k, then the degrees $m \geq L$ of its polynomial solutions (3) satisfy to inequality

$$m \geq \min\{M_{ki}\colon i = \overline{0,N},\ k \neq i\}.$$

For linear differential equation

$$\sum_{i=o}^{n} B_i(z) w^{(\ell_i)} = 0,\ n \geq 1, \tag{2.2}$$

where $0 \leq \ell_o < \ell_1 < \ldots < \ell_n$, $b_i - b_j = \ell_i - \ell_j$, $i=\overline{0,n}$, $j=\overline{0,n}$, $j \neq i$, the partial case of which is well-known Euler's equation [85]

$$\sum_{\ell=o}^{n} a_\ell z^\ell w^{(\ell)} = 0,\quad a_\ell = \text{const.},$$

on the base of theorem 2.5 we get

Consequence 2.3. The degrees m of polynomial solutions of linear differential equation (2.2) are the roots of equation

$$\sum_{i=o}^{\delta} \ell_i! \ \binom{m}{\ell_i} \beta_i = 0,$$

where $\delta = n$ for $m \geq \ell_n$, $\delta = r$ for $\ell_r \leq m \leq \ell_{r+1}$, $r \in \{0,1,\dots,n-1\}$.

Analogous result for $m \geq \ell_n$ was got in [49].

§3. THE COEFFICIENTS OF HIGHEST TERM OF POLYNOMIAL SOLUTION

By method analogous to the method of proof of theorem 2.4 and 2.5 following theorem, with the help of which the coefficients α_m of highest term of polynomial solution (3) of degree m of differential equation (1) are found, is proved [63,72,74].

Theorem 3.1. Let following correlations take a place:

1) $x_\tau < x_p = x_{p+1} = \dots = x_{p+s} = d_p > x_\eta$, $\tau=\overline{0,p-1}$, $0 \leq p \leq N$, $0 \leq s \leq N-p$, $\eta=\overline{p+s+1,N}$;
2) $b_p - m_p = \underline{b_{p+1} - m_{p+1}} = \dots = b_{p+\lambda} - m_{p+\lambda} > b_{p+j} - m_{p+j}$, $0 \leq \lambda \leq s$, $j=\overline{\lambda+1,s}$.

Then following statements are right:

a) for $p=0$, $0 \leq s < N$, $M_{o,s+1}=M_{o,s+2}=\dots=M_{or}=M_o$, $s+1 \leq r \leq N$;
b) for $0<p\leq N$, $s=N-p$, $M_{po}=M_{p1}=\dots=M_{p\ell}=M_p$, $0 \leq \ell \leq p-1$;
c) for $0<p<N$, $0 \leq s < N-p$, $M_{po}=M_{p1}=\dots=M_{p\ell}=M_p$, $0 \leq \ell \leq p-1$;
d) for $0<p<N$, $0 \leq s < N-p$, $M_{p,p+s+1}=M_{p,p+s+2}=\dots=M_{pr}=M_p$, $p+s+1 \leq r \leq N$;
e) for $0<p<N$, $0 \leq s < N-p$, $M_{po}=M_{p1}=\dots=M_{p\ell}=M_p$, $M_{p,p+s+1}=M_{p,p+s+2}=\dots=M_{pr}=M_p$, $0 \leq \ell \leq p-1$, $p+s+1 \leq r \leq N$.

The coefficient α_m for highest term of polynomial solution (3) of differential equation (1) of the degree:

a) $m = \mathcal{B}_{os} \geq L$;
b) $m = \mathcal{U}_p \geq L$;
c) $m = \mathcal{U}_p \geq L$, $m > \mathcal{B}_{ps}$;
d) $m = \mathcal{B}_{ps} \geq L$, $m < \mathcal{U}_p$;
e) $m = \mathcal{U}_p = \mathcal{B}_{ps} \geq L$

is accordingly the root of equation

a) $\alpha_m^{d_o} \sum_{h=o}^{\lambda} \tilde{K}_h(m) + \sum_{\eta=s+1}^{r} K_\eta(m,\alpha_m) = 0$;

b) and c) $\sum_{\tau=0}^{\ell} K_\tau(m,\alpha_m) + \alpha_m^{d_p} \sum_{h=1}^{\lambda} \tilde{K}_{p+h}(m) = 0;$

d) $\alpha_m^{d_p} \sum_{h=0}^{\lambda} \tilde{K}_{p+h}(m) + \sum_{\eta=p+s+1}^{r} K_\eta(m,\alpha_m) = 0;$

e) $\sum_{\tau=0}^{\ell} K_\tau(m,\alpha_m) + \alpha_m^{d_p} \sum_{h=0}^{\lambda} \tilde{K}_{p+h}(m) + \sum_{\eta=p+s+1}^{r} K_\eta(m,\alpha_m) = 0.$

§4. ENTIRE SOLUTIONS OF IRREDUCIBLE PAINLEVE'S EQUATIONS

Theorems 2.1-2.3 in totality with theorems 2.4, 2.5 and 3.1 permit to establish the presence of entire solutions both transcendent and algebraic (polynomial). We show this on the example of the equations of Painleve's type, which are investigated manysidely in last decades, including by Belorussian mathematicians N.A.Lukashevich and by this pupil V.I.Gromak [86,87], by I.A.Jablonskij and by his pupil I.P.Martynov [88,89,99] and others of N.P.Erugine's school [83,90-92] too.

The first irreducible Painleve's equation

$$w'' - 6w^2 - z = 0 \qquad \text{(P-1)}$$

has one dominated term and so among there is no either polynomial or enitre transcendent solutions. It is conformed with known fact that all solutions of equation (P-1) have infinite number of poles condensating to $z = \infty$ [83, p. 56; 93].

The second irreducible Painleve's equation

$$w'' - 2w^3 - zw - \alpha = 0, \quad \alpha = \text{const.}, \qquad \text{(P-2)}$$

has not entire transcendent solutions and has only one polynomial solution $w = 0$ for $\alpha = 0$.

It was the simplest cases of Painleve's equations is resolving the problem of entire solutions. The matter of the rest equations of the second and the third orders of Painleve's type with polynomial and entire transcendent solutions stand in following way.

<u>Theorem 4.1</u>. The third Painleve's equation

$$zww'' - z(w')^2 + ww' - \gamma zw^4 - \alpha w^3 - \beta w + \delta z = 0, \qquad \text{(P-3)}$$

where $\alpha,\beta,\gamma,\delta$ are some constants, can have only following polynomial solutions:

a) $w = 0$ for $\delta = 0$;

b) $w = C$ and $w = Cz$, C=const., for $\alpha = \beta = \gamma = \delta = 0$;

c) $w = \pm\sqrt{\beta/\alpha}$ for $\alpha\gamma\delta \neq 0$, $\alpha^2\delta + \beta^2\gamma = 0$ or for $\gamma = \delta = 0$, $\alpha\beta \neq 0$;

d) $w = a_i$, where a_i are the roots of equation $\gamma a^4 + \delta = 0$, for $\alpha = \beta = 0$, $\gamma\delta \neq 0$;

e) $w = -(\delta/\beta)z$ for $\alpha = \gamma = 0$, $\beta\delta \neq 0$, $\delta + \beta^2 \neq 0$;

f) $w = \beta z + C$, C=const., for $\alpha = \gamma = 0$, $\beta\delta \neq 0$, $\delta + \beta^2 = 0$;

g) has polynomial solution of degree $m \geq 2$ for $\alpha = \gamma = 0$.

Theorem 4.2. Only

a) $w = -2z$ for $\alpha = 0$, $\beta = -2$;

b) $w = -(2/3)z$ for $\alpha = 0$, $\beta = -2/9$;

c) $w = 0$ for $\beta = 0$

are polynomial solutions of the fourth Painleve's equation

$$2ww'' - (w')^2 - 3w^4 - 8zw^3 - 4(z^2-\alpha)w^2 - 2\beta = 0, \qquad \text{(P-4)}$$

where α and β are constants.

Equation (P-4) has not entire transcendent solutions i.e. it contains one dominated term $-3w^4$. Therefore all possible entire solutions of equation (P-4) are pointed in theorem 4.2.

Theorem 4.3. The fifth Painleve's equation

$$2z^2w^2w'' - 2z^2ww' - 3z^2w(w')^2 + z^2(w')^2 + 2zw^2w' - 2zww' - $$
$$- 2\alpha w^5 + 6\alpha w^4 - 2(\delta z^2+\gamma z+3\alpha+\beta)w^3 - 2(\delta z^2-\gamma z-\alpha-3\beta)w^2 - \qquad \text{(P-5)}$$
$$- 6\beta w + 2\beta = 0,$$

where α, β, γ, δ are some constants, can have only following polynomial solutions:

a) $w = 0$ for $\beta = 0$;

b) $w = C$, C=const., for $\alpha = \beta = \gamma = \delta = 0$;

c) $w = 1$ for $\delta = 0$, $|\alpha| + |\beta| + |\gamma| \neq 0$;

d) $w = -1$ for $\gamma = 0$, $\alpha + \beta = 0$, $\delta \neq 0$;

e) $w = -2(\delta/\gamma)z + 1$ for $\alpha\gamma\delta \neq 0$, $\beta = -1/2$, $4\alpha\delta + \gamma^2 = 0$;

f) $w = az + b$, where $a^2 + 2\delta = 0$, $b^2 + 2\beta = 0$, for $\alpha = 1/2$, $\delta \neq 0$, $\gamma^4 + 64\delta^2 + 16\beta^2\delta^2 - 8\beta\delta\gamma^2 + 64\beta\delta^2 + 16\gamma^2\delta = 0$;

g) of degree $m \geq 1$ such that $m^2 + 2\beta = 0$ for $\alpha = \gamma = \delta = 0, \beta \neq 0$.

Theorem 4.4. The fifth Painleve's equation (P-5) can have entire transcendent solutions only $\Omega = 1$ (for $\alpha = 0$, $\delta \neq 0$) or $\Omega = 1/2$ (for $\alpha = \delta = 0$, $\gamma \neq 0$) orders; if $\alpha \neq 0$ or $\alpha = \gamma = \delta = 0$, then (P-5) has not entire transcendent solutions.

The example of entire transcendent solution of $\Omega = 1$ order was built for (P-5) by N.A.Lukashevich [86, p. 119]: if $\alpha = \beta = 0$, $\gamma^2 + 2\delta = 0$, $\delta \neq 0$ then equation (P-5) has one-parametric family of solutions $w = C \exp \gamma z$, C=const. Entire transcendent solution $\Omega = 1/2$ order for (P-5) is not built on today.

Theorem 4.5. The sixth Painleve's equation

$$2z^2(z-1)^2w^3w'' - 2z^2(z-1)^2(z+1)w^2w'' + 2z^2(z-1)^2ww''$$
$$- 3z^2(z-1)^2w^2(w')^2 + 2z^2(z-1)^2(z+1)w(w')^2 - z^3(z-1)^2(w')^2 +$$
$$+ 2z(z-1)(2z-1)w^3w' - 2z(z-1)(z^2+2z-1)w^2w' + 2z^3(z-1)ww' -$$
$$- 2\alpha w^6 + 4\alpha(z+1)w^5 - 2[(\alpha+\beta)z^2 + (4\alpha+\beta+\gamma+\delta)z + \alpha - \gamma]w^4 +$$
$$+ 4[(\alpha+\beta+\gamma+\delta)z + \alpha + \beta - \gamma - \delta]zw^3 - 2z[(\beta+\gamma)z^2 + \qquad \text{(P-6)}$$
$$+ (\alpha+4\beta-\gamma+\delta)z + \beta - \delta]w^2 + 4\beta z^2(z+1)w - 2\beta z^2 = 0,$$

where α, β, γ, δ are constants, can have polynomial solutions:

a) of degree $m \geq 2$ for $\alpha = 0$, such that $m^2 - 2m + \delta = 0$;
b) of degree $m = 1$;
c) of degree $m = 0$ for $\beta = 0$ or for $\beta + \gamma = 0$, $\alpha + \delta \neq 0$, or for $\alpha \neq 0$, $\beta = \gamma = 0$, $\alpha + \delta = 0$, moreover for $\alpha = \beta = \gamma = \delta = 0$ there is one-parametric family of solutions $w = C$, C=const., and for $\beta = 0$ there is the solution $w = 0$.

Theorem 4.6. The sixth Painleve's equation (P-6) has not entire transcendent solutions.

The absence of entire transcendent solutions for (P-6) point on two following peculiarity:

1) all entire solutions of equation (P-6) are polynomials and possible cases of its existence are pointed in theorem 4.5;

2) it is known [94] that solutions on the fifth Painleve's equation (P-5) can be got from the solutions of the sixth Painleve's equation (P-6) but in doing so the fifth Painleve's equation has

entire transcendent solutions and the sixth Painleve's equation has not them.

As for the absence of movable critical singular points of solutions (Painleve's type) I.P.Martynov considered differential equation of the third order [89,95]

$$\begin{aligned} & w^2w'w''' + w^4w''' - (\nu-1)\nu^{-1}w^2(w'')^2 - a_1w(w')^2w'' + \\ & + [a_1 - a + 4(\nu-1)\nu^{-1}]w^3w'w'' + aw^5w'' - b_1(w')^4 + \\ & + (b_1-b)w^2(w')^3 + [b - c - 4(\nu-1)\nu^{-1}]w^4(w')^2 - \\ & - (c-d)w^6w' + dw^8 = 0, \end{aligned} \tag{4.1}$$

where ν is the integer number different from zero; a, b, c, a_1, b_1 are constants.

Theorem 4.7. Equation (4.1) can have only following polynomial solutions (3):

a) of degree $m = 2$ for $a = c = d = 0$, $b = a_1 + 2b_1$, $b = 4(\nu-1)\nu^{-1}$, $a_1 + b_1 \neq 0$ or for $a \neq 0$, $c = d = 0$, $a + 2b = 8(\nu-1)\nu^{-1}$;

b) of degree $m \geq 3$ for fulfilment even though of conditions:

$$a = c = d = 0,\ b = 4(\nu-1)\nu^{-1},\ (b+a_1)(b-b_1) \neq 0; \tag{4.2}$$

$$a = c = d = 0,\ b = 4(\nu-1)\nu^{-1},\ b = b_1 \neq -a_1; \tag{4.3}$$

$$a = c = d = 0,\ b = 4(\nu-1)\nu^{-1},\ b = -a_1 \neq b_1; \tag{4.4}$$

$$c = d = 0,\ b \neq 4(\nu-1)\nu^{-1},\ a \neq 0 \tag{4.5}$$

moreover for conditions (4.2)-(4.4) the number m must be the root of equation

$$(a_1 + b_1 - 1)m^2 - (b + a_1 - 3)m - 2 = 0,$$

and for condition (4.5) m must be the root of equation

$$[(a + b)\nu - 4(\nu-1)]m - a\nu = 0.$$

Besides there are trivial polynomial solutions: $w = 0$; for $d = 0$ there is $w = C_1$; for $b = b_1 = c = d = 0$, $\nu = 1$ there is $w = C_1z + C_2$; for $a = a_1 = b = b_1 = c = d = 0$, $\nu = 1$ there is $w = C_1z^2 + C_2z + C_3$, where C_1, C_2, C_3 are constants.

If even though one of conditions: 1) $d \neq 0$; 2) $d = 0$, $c \neq 0$; 3) $d = c = 0$, $4\nu \neq 4 - (a + b)$; 4) $a = b = c = d = 0$, $\nu = 1$,

$a_1 + b_1 \neq 1$ are fulfiled, then equation (4.1) has not entire transcendent solutions.

On the base of theorem 4.7 the existence of entire solutions for the equations of the third order (4.1), when it is Painleve's type, is established [89].

§5. THE QUANTITY OF POLYNOMIAL SOLUTIONS OF DIFFERENT DEGREES

A.Z.Samujlov [33] pointed interesting regularity for algebraic differential equation of type

$$A(x)y^{(m)} = \sum_{i=0}^{n} B_\nu(x) y^{\nu_i},$$

which was that the quantity of different degrees, that have polynomial solutions, depend on the quantity of the terms in this equation. The generalizations was given by him [34], by L.G.Oreshchenko [35], by S.I.Kishel [62] for special types of equations, and for algebraic equation of general type (1) they was given in [73,74].

In given paragraph polynomial solutions of differential equation (1) we'll find in the form

$$w_k = \sum_{\ell=0}^{m_k} \alpha_{m_{k_\ell}} z^{m_k-\ell}, \quad \alpha_{m_{k_0}} \neq 0, \; k=\overline{0,p}, \tag{5.1}$$

counting $m_0 < m_1 < \ldots < m_p$.

Following statement contain main sum.

<u>Theorem 5.1</u>. If in algebraic differential equation (1) the terms are the situated such that

$$\begin{aligned} &L_0 = L_1 = L_2 = \ldots = L_{h_0} < L_{h_0+1} = L_{h_0+2} = \ldots = \\ &= L_{h_0+h_1} < L_{h_0+h_1+1} = L_{h_0+h_1+2} = \ldots = L_{h_0+h_1+h_2} < \ldots < \\ &< L_{1+\sum_{j=0}^{t-1} h_j} = L_{2+\sum_{j=0}^{t-1} h_j} = \ldots = L_{\sum_{j=0}^{t} h_j}, \end{aligned} \tag{5.2}$$

where $\sum_{j=0}^{t} h_j = N$, $0 \le h_0 \le N$, $0 \le h_{\varepsilon+1} \le N - \sum_{\gamma=0}^{\varepsilon} h_\gamma$, $\varepsilon=\overline{0,t-1}$, and for x_i, $i = 0, \sum_{j=0}^{\tau} h_j$, $\tau=\overline{0,t}$, such that

$$x_{\mu_o^\delta} = x_{\mu_1^\delta} = \ldots = x_{\mu_{\xi_\delta}^\delta}, \quad \delta=\overline{0,\lambda}, \quad 0 \le \lambda < \sum_{j=o}^{h} h_j, \tag{5.3}$$

for any natural numbers n_ψ^δ, $\psi=\overline{0,\eta_\delta}$,

$$L_\delta \le n_o^\delta \le n_1^\delta \le \ldots \le n_{\eta_\delta}^\delta, \quad \delta=\overline{0,\lambda}, \tag{5.4}$$

where $L_\delta = \max\{L_{\mu_\psi^\delta}: \psi=\overline{0,\eta_\delta}\}$, inequantities

$$\begin{vmatrix} \tilde{K}^*_{\mu_o^\delta}(n_o^\delta) & \tilde{K}^*_{\mu_1^\delta}(n_1^\delta) & \ldots & \tilde{K}^*_{\mu_{\eta_\delta}^\delta}(n_o^\delta) \\ \tilde{K}^*_{\mu_o^\delta}(n_1^\delta) & \tilde{K}^*_{\mu_1^\delta}(n_1^\delta) & \ldots & \tilde{K}^*_{\mu_{\eta_\delta}^\delta}(n_1^\delta) \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{K}^*_{\mu_o^\delta}(n_{\eta_\delta}^\delta) & \tilde{K}^*_{\mu_1^\delta}(n_{\eta_\delta}^\delta) & \ldots & \tilde{K}^*_{\mu_{\eta_\delta}^\delta}(n_{\eta_\delta}^\delta) \end{vmatrix} \ne 0 \tag{5.5}$$

are fulfil for $\delta=\overline{0,\lambda}$, then for all $\tau=\overline{0,t}$ algebraic differential equation (1) can have non-trivial polynomial solution (5.1) of different degrees m_k, $k=\overline{0,p}$, such that

$$L_{\sum_{j=o}^{\tau} h_j} \le m_p < L_{1+\sum_{j=o}^{\tau} h_j} \tag{5.6}$$

where $L_{1+\sum_{j=o}^{\tau} h_j} = +\infty$ for $\sum_{j=o}^{\tau} h_j = N$, no more then $\sum_{j=o}^{\tau} h_j$.

More obvious theorems are true as the consequences.

<u>Theorem 5.2</u>. For conditions (5.2)-(5.5) algebraic differential equation (1) has no more then N non-trivial polynomial solutions of different degrees.

<u>Theorem 5.3</u>. If in algebraic differential equation (1) the terms are situated such that the correlation (5.2) takes a place and $x_i \ne x_j$, $i=\overline{0,N}$, $j=\overline{0,N}$, $j \ne i$, then for all $\tau=\overline{0,t}$ equation (1) has no more $\sum_{j=o}^{\tau} h_j$ non-trivial polynomial solutions (5.1) of different degrees m_k, satisfying to the condition (5.6).

For example, algebraic differential equation

$$\sum_{i=o}^{n} A_i(z) \prod_{j=o}^{h} \prod_{\tau=o}^{M_j} \{w^{(r_{\tau j})}\}^{p_{\tau j}\alpha_{ji}} = \sum_{k=o}^{m} B_{p_k}(z) w^{\gamma_k} \prod_{\lambda=o}^{M} \{w^{(r_\lambda)}\}^{s_\lambda},$$

for which
$$\sum_{j=o}^{h} \sum_{\tau=o}^{M_j} p_{\tau j}\alpha_{ji} < \sum_{j=o}^{h} \sum_{\tau=o}^{M_j} p_{\tau j}\alpha_{j,\tau+1},$$
has no more then n + m + 1 non-trivial polynomial solutions of different degrees if under the existence in this equation the terms such that
$$\sum_{j=o}^{h} \sum_{\tau=o}^{M_j} p_{\tau j}\alpha_{ji} = \gamma_k + \sum_{\lambda=o}^{M} S_\lambda, \ i\in\{0,1,\dots,n\}, \ j\in\{0,1,\dots,m\},$$
inequality
$$\max\{r_{\tau j}: j=\overline{0,h}, \ \tau=\overline{0,M_j}\} \geq \max\{r_\lambda: \lambda=\overline{0,M}\} = r_\sigma$$
is fulfiled and
$$\sum_{j=o}^{h} \sum_{\tau=o}^{M_j} \delta_{\tau j}p_{\tau j}\alpha_{ji} > p_\sigma,$$
where $\delta_{\tau j} = 1$ if $r_{\tau j} \geq r_\sigma$, or $\delta_{\tau j} = 0$ if $r_{\tau j} < r_\sigma$.

R E F E R E N C E S*

[1] Gorbuzov V.N.: Polinomial solutions of algebraic ordinary differential equations I, Bulletin de la societé des mathem. et des informat. de Macedonie 17(XLIII), Skopje, 1993, 5-20

ПОЛИНОМНИ РЕШЕНИЈА НА АЛГЕБАРСКИ ОБИЧНИ ДИФЕРЕНЦИЈАЛНИ РАВЕНКИ. II

В.Н. Горбузов

Р е з и м е

Во овој труд е разгледуван граничен метод за дефинирање на степените на полиномните решенија на алгебарски диференцијални равенки и цели решенија на равенките на Peinleve. Најдени се коефициентите на највисокиот израз на полиномните решенија и квантитетот на полиномните решенија од различни степени на алгебарските диференцијални равенки.

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* Note of the editor: The reference in this paper is the same as in the paper [1]