Математички Билтен
19 (XLV)
1995 (5–16)
Скопје, Македонија

ISSN 0351–336X

# POLYNOMIAL SOLUTIONS OF ALGEBRAIC ORDINARY DIFFERENTIAL EQUATIONS. III

V. N Gorbuzov

**Abstract**

The building of polynomial solutions on the whole of algebraic differential equations are considered.

## & 6. The building of polynomial solutions on the whole

The finding of solutions–polynomials of equation (1) that indentity vanish definite structural forming of equation (1) is putting into the base of reasonings. For this purpose we write algebraic differential equation in the form

$$T\left(z,\,\omega,\,\omega',\,\ldots,\,\omega^{(\beta)}\right)+\psi^{l}\left(z,\,\omega,\,\omega',\ldots,\,\omega^{(\gamma)}\right)\times \\ \times\Big\{A_j(z)\phi_j^{\delta}\left(z,\omega,\omega',\ldots,\omega^{(\rho_j)}\right)+A_r(z)\phi_r^{\delta}\left(z,\omega,\omega',\ldots,\omega^{(\rho_r)}\right)\Big\}=0. \tag{6.1}$$

where $T$, $\psi$, $\phi_k$ and $A_k$ $(k\in\{r,\,j\})$ are the polynomial of their arguments; $\beta$, $l$, $\gamma$ and $\rho_k$ are non–negative integer numbers; $a_r=\deg A_r(z)\geq a_j=\deg A_j(z)$, $\delta$ is natural number, moreover $(a_r-a_j)\delta^{-1}=s$ is non–negative integer number.

Mark
$$\mathcal{G}(z)=\left[\left(-A_r(z)/A_j(z)\right)^{1/\delta}\right]\,,$$

---

*Matematics subject classifications* (1980): **34**, A 05,
*Keywords and phrases*: Ordinary equation, polynomial solution

where the symbol [ ] means polynomial part of expansion on decreasing degrees $z$. Then $\deg \mathcal{G}(z) = g$.

We define the polynomial $Q(z)$ from the correlation

$$A_r(z) = -A_j(z)\mathcal{G}^\delta(z) - Q(z)\,. \tag{6.2}$$

Replacing in (6.1) the coefficient $A_r(z)$ by $A_j(z)$, $\mathcal{G}(z)$ and $Q(z)$ on the base of (6.2) we form the equality

$$F\left(z, \omega, \omega', \ldots, \omega^{(\alpha)}\right) = A_j(z)\psi^l\left(z, \omega, \omega', \ldots, \omega^{(\gamma)}\right) *$$
$$* \left\{\phi_j^\delta\left(z, \omega, \omega', \ldots, \omega^{(\rho_j)}\right) - \mathcal{G}^\delta(z)\phi_r^\delta\left(z, \omega, \omega', \ldots, \omega^{(\rho_r)}\right)\right\}.$$

where

$$F\left(z, \omega, \omega', \ldots, \omega^{(\alpha)}\right) = Q(z)\psi^l\left(z, \omega, \omega', \ldots, \omega^{(\gamma)}\right) *$$
$$* \phi_j^\delta\left(z, \omega, \omega', \ldots, \omega^{(\rho_j)}\right) - \mathcal{G}^\delta(z)\phi_r^\delta\left(z, \omega, \omega', \ldots, \omega^{(\rho_r)}\right\}.$$

Further it is expediently to input in addition following conditional meanings:

$$F\left(z, \omega(z), \omega'(z), \ldots, \omega^{(\alpha)}(z)\right) = F\big(z; \omega(z)\big).$$
$$\psi\left(z, \omega(z), \omega'(z), \ldots, \omega^{(\gamma)}(z)\right) = \psi\big(z; \omega(z)\big),$$
$$\phi_k\left(z, \omega(z), \omega'(z), \ldots, \omega^{(\rho_r)}(z)\right) = \phi_k\big(z; \omega(z)\big),$$
$$f(m) = \deg F\big(z; \omega(z)\big), \qquad \psi(m) = \deg \psi\left(z; \omega(z)\right),$$
$$\varphi_k(m) = \deg \phi_k\big(z; \omega(z)\big),$$

where $\omega(z)$ is polynomial of degree $\deg \omega(z) = m$.

Following two theorems are the essence of method [96].

*Theorem* **6.1**. For the polynomial $\omega = \omega(z)$ of degree $m$ to be the solution of equation (6.1) fulfilment even though one of the conditions:

$$\varphi_j(m) = \varphi_r(m) + g, \quad f(m) \le a_j + l\psi(m) + \delta\varphi_j(m)\,. \tag{6.3}$$

or

$$\varphi_j(m) \ge \varphi_r(m) + g, \quad f(m) = a_j + l\,\psi(m) + \delta\left\{g + \varphi_r(m)\right\}$$

or

$$\varphi_j(m) \le \varphi_r(m) + g, \quad f(m) = a_j + l\psi(m) + \delta\left\{g + \varphi_r(m)\right\}.$$

*Theorem* **6.2**. Equation (6.1) can have polynomial solution $\omega = \omega(z)$ of degree $m$, satisfying to the correlations (6.3), if this polynomial is a solution of the equation

$$\psi\left(z, \omega, \omega', \ldots, \omega^{(\gamma)}\right) = 0 \tag{6.4}$$

or a solution even though one of the equations

$$\phi_j\left(z, \omega, \omega', \dots, \omega^{(\rho_j)}\right) - \varepsilon_t G(z)\phi_r\left(z, \omega, \omega', \dots, \omega^{(\rho_r)}\right) = \varepsilon_t P(z), \quad (6.5)$$

$t = \overline{1, \delta}$,

where $\varepsilon_t$ are the roots of the equation $\varepsilon^\delta = 1$, $P(z)$ is some polynomial of degree $\deg P(z) = \rho$, defined by the correlation

$$\rho \le f(m) - a_j - l\psi(m) - (\delta - 1)\left\{g + \varphi_r(m)\right\},$$

and for the condition

$$f(m) < a_j + l\psi(m) + \delta\varphi_j(m)$$

defined by the correlation

$$\rho = f(m) - a_j - l\psi(m) - (\delta - 1)\left\{g + \varphi_r(m)\right\}$$

moreover, if $\rho < 0$ then $P(z) = 0$.

If the polynomial $\omega = \omega(z)$ of degree $m$ is the solution of the equation (6.4), then for this polynomial to be the solution of the equation (6.1) is necessary and sufficient the indentity $T\big(z; \omega(z)\big) \equiv 0$, where $T\big(z; \omega(z)\big) = T\left(z, \omega(z), \omega'(z), \dots, \omega^{(\beta)}(z)\right)$ fulfils.

If the polynomial $\omega = \omega(z)$ of degree $m$ is the solution even though one of the equations (6.5) for $P(z) \equiv 0$ then for the polynomial to be the solution of the equation (6.1) the fulfilment of identity $F\big(z; \omega(z)\big) \equiv 0$ is necessary and sufficient.

Therefore the finding of polynomial solutions (3) of algebraic differential equation (1) can be realized by the means of consequent fulfilment of the following steps.

**1**. The leading of the equation (1) to the type (6.1), for which the correlations (6.3) are right, i.e. to the equation for which exactly two composing have identical the largest degrees for the substitution of the polynomial $\omega = \omega(z)$.

**2**. The finding of the famility of polynomials of degree $m$, satisfying to (6.3), on the base of equations (6.4) and (6.5) of theorem **6.2**.

**3**. The picking out from the all multitude polynomial of getting structure those that are the solution of initial differential equation (1).

The leading of equation (1) to the form (6.1) for the condition (6.3) can be realized on the base of arbitrary groupping of the terms of equation (1) if only in the consequence the correlation (6.3) fulfil. In some cases as following from theoretical or practical interests it is worth while to lead differential equation (1) to special forms. Point some classes of such equations.

If $l = 0$ or $\psi\left(z,\, \omega,\, \omega',\, \ldots,\, \omega^{(\gamma)}\right) \equiv 1$, then equation (6.1) can be wrote in the form

$$T\left(z,\, \omega,\, \omega',\, \ldots,\, \omega^{(\beta)}\right) + A_r(z)\phi_r^{\delta}\left(z,\, \omega,\, \omega',\, \ldots,\, \omega^{(\rho_r)}\right) + \\ + A_j(z)\phi_j^{\delta}\left(z,\, \omega,\, \omega',\, \ldots,\, \omega^{(\rho_j)}\right) = 0\,, \tag{6.6}$$

and, as a consequence of theorem **6.2**, is just

*Theorem* **6.3**. Algebraic differential equation (6.6) can have polynomial solution $\omega = \omega(z)$ of degree $m$ satisfying to the conditions

$$t(m) < a_r + \delta\varphi_r(m) = a_j + \delta\varphi_j(m)\,,$$

where $t(m) = \deg T\big(z;\, \omega(z)\big)$, if this polynomial is a solution even though one of the equations (6.5), where $P(z)$ is some polynomial of degree $\rho = f(m) - a_j - (\delta - 1)\varphi_j(m)$, moreover, if $\rho < 0$ then $P(z) \equiv 0$.

If $\phi_r\left(z,\, \omega,\, \omega',\, \ldots,\, \omega^{(\rho_r)}\right) \equiv 1$ then equation (6.1) for the condition (6.3) can be wrote in the form

$$r\left(z,\, \omega,\, \omega',\, \ldots,\, \omega^{(\beta)}\right) + \psi^{l}\left(z,\, \omega,\, \omega',\, \ldots,\, \omega^{(\gamma)}\right) \times \\ \times \left\{A_r(z) + A_j(z)\phi_j^{\delta}\left(z,\, \omega,\, \omega',\, \ldots,\, \omega^{(\rho_j)}\right)\right\} = 0\,, \tag{6.7}$$

and theorem **6.4** is true.

*Theorem* **6.4**. Algebraic differential equation (6.7) can have polynomial solution $\omega = \omega(z)$ of degree $m$ satisfying to the correlations

$$t(m) < a_r + l\psi(m), \qquad a_j = a_r - \delta\varphi_j(m)\,, \tag{6.8}$$

if this polynomial is the solution of equation (6.4) or the solution even though of one of equations

$$\phi_j\left(z,\, \omega,\, \omega',\, \ldots,\, \omega^{(\rho_j)}\right) = \varepsilon_t \mathcal{G}(z) + \varepsilon_t P(z), \quad t = \overline{1,\, \delta}\,,$$

where $P(z)$ is polynomial of degree $\rho = f(m) - a_j - l\psi(m) - (\delta - 1)\varphi_j(m)$, moreover, if $\rho < 0$ then $P(z) \equiv 0$.

If $\psi\left(z,\, \omega,\, \omega',\, \ldots,\, \omega^{(\gamma)}\right) = \phi_j\left(z,\, \omega,\, \omega',\, \ldots,\, \omega^{(\rho_j)}\right)$ then equation (6.7) will take the form

$$T\left(z,\, \omega,\, \omega',\, \ldots,\, \omega^{(\beta)}\right) + A_r(z)\psi^{l}\left(z,\, \omega,\, \omega',\, \ldots,\, \omega^{(\gamma)}\right) + \\ + A_j(z)\psi^{l+\delta}\left(z,\, \omega,\, \omega',\, \ldots,\, \omega^{(\gamma)}\right) = 0\,, \tag{6.9}$$

and as a consequence of theorem **6.4** theorem **6.5** is true.

*Theorem* **6.5**. Algebraic differential equation (6.9) can have polynomial solution $\omega = \omega(z)$ of degree $m$ satisfyng to the correlations (6.8) if this polynomial is the solution of equation (6.4) or the solution even thought of one of the equations

$$\psi\left(z, \omega, \omega', \ldots, \omega^{(\gamma)}\right) = \varepsilon_t \mathcal{G}(z) + \varepsilon_t P(z), \quad t = \overline{1, \delta},$$

where $P(z)$ is polynomial of the degree $\rho = f(m) - a_j - (l + \delta - 1)\,\psi(m)$, moreover, if $\rho < 0$ then $P(z) \equiv 0$.

If in the equation (6.8) the constituent $\phi_j\left(z, \omega, \omega', \ldots, \omega^{(\rho_j)}\right) \equiv \omega^{(з)}$ then it will have the form

$$T\left(z, \omega, \omega', \ldots, \omega^{(\beta)}\right) + \psi^l\left(z, \omega, \omega', \ldots, \omega^{(\gamma)}\right) \times \\ \times \left\{A_r(z) + A_j(z)\left(\omega^{(з)}\right)^{\delta}\right\} = 0, \tag{6.10}$$

and as a consequence of the theorem **6.4** (also as of theorem **6.5** too) theorem **6.6** is true.

*Theorem* **6.6**. Algebraic differential equation (6.10) can have polynomial solution $\omega = \omega(z)$ of degree $m$ satisfying to the correlations $t(m) < a_r + l\psi(m)$, $a_j = a_r - \delta(m - з)$, if this polynomial $\omega = \omega(z)$ is the solution of equation (6.4) or this one have the structure

$$\omega = \varepsilon_t \mathcal{J}^з \mathcal{G}(z) + D(z), \quad t = \overline{1, \delta}, \tag{6.11}$$

where $\mathcal{J}^з$ is the operator such that $\mathcal{J}^з z^k = \{(k+1)_з\}^{-1} z^{k+з}$, $(k+1)_з$ is Pochhammer's symbol, $D(z)$ is some polynomial od degree $\deg D(z) = d$, defined by the correlation $d = \rho + з$ for $\rho \geq 0$, $d \leq з - 1$ for $\rho < 0$, the number $\rho = f(m) - a_j - l\psi(m) - (\delta - 1)g$. If $\psi\left(z, \omega, \omega', \ldots, \omega^{(\gamma)}\right) = \omega^{(з)}$ then the equation (6.10) will have the form

$$T\left(z, \omega, \omega', \ldots, \omega^{(\beta)}\right) + A_r(z)\left\{\omega^{(з)}\right\}^l + A_j(z)\left\{\omega^{(з)}\right\}^{l+\delta} = 0, \tag{6.12}$$

and as a consequence of the theorem **6.6** theorem **6.7** is true.

*Theorem* **6.7**. Polynomial solutions $\omega = \omega(z)$ of degree $m = з + g$ such that $t(m) < a_r + l(m - з)$, $a_j = a_r - \delta(m - з)$ of algebraic differential equation (6.12) have the structure (6.11) where $D(z)$ is some polynomial of degree $m$ defining by the correlation $d = з + \rho$ for $\rho \geq 0$, $d \leq з - 1$ for $\rho < 0$, the number $, \rho = f(з + g) - a_j - (l + \delta - 1)g$.

Particularly from theorem **6.7** known E. D. Reinville's theorem [1; 97, p. 44; 98, p. 255] about the presence and the structure of polynomial solutions of algebraic Riccati's eqauation follows that (as pointed in introduction) become point of departure in the creation of the method described in this paragraph.

## & 7. Maximal number of polynomial solutions of definite structure

In & **6** the method of the finding of all the polynomial's of degree $m$ familiity was considered in which polynomial splutions of degree $m$ of algebraic differential equation (1) are contained. The following problem is putting: to point out the conditions for which all polynomials of famility will be the solution of equation (1) [25, 30, 37, 71, 81].

Consider the famility of polynomials

$$\omega = \varepsilon_t \mathcal{G}(z), \qquad t = \overline{1,\, \delta}\,, \tag{7.1}$$

where $\mathcal{G}(z)$ is the polynomial, $\varepsilon_t$ are the roots of the equation $\varepsilon^\delta = 1$ such that $\varepsilon_t = \varepsilon^t$.

The terms of algebraic differential equation (1), in dependence of their dimensions, we put such that

$$\begin{array}{llll} \kappa_0 = \kappa_{\rho_0} + \xi_{\rho_0}\delta, & \rho_0 = \overline{0,\, \lambda_1}, & 0 \le \lambda_1 \le N; & \\ \kappa_{\lambda_1+1} = \kappa_{\rho_1} + \xi_{\rho_1}\delta, & \rho_1 = \overline{\lambda_1+1,\, \lambda_2}, & \lambda_1 + 1 \le \lambda_2 \le N; & \\ \cdots \quad \cdots \quad \cdots & \cdots \quad \cdots \quad \cdots & \cdots \quad \cdots \quad \cdots & \\ \kappa_{\lambda_{h-1}+1} = \kappa_{\rho_{h-1}} + \xi_{\rho_{h-1}}\delta, & \rho_{h-1} = \overline{\lambda_{h-1}+1,\, \lambda_h}, & \lambda_h = N, & \end{array} \tag{7.2}$$

where $\xi_{\rho_j} = \overline{0,\, h-1})$ are integer numbers, $h \le \delta$.

*Theorem* **7.1**. For all polynomials of famility (7.1) to be the solutions of algebraic differential equation (1) it is necessary and sufficient for the polynomial $\mathcal{G}(z)$ to be the solution of the system of equations

$$\sum_{\tau=\lambda_{\alpha-1}+1}^{\lambda_\alpha} B_T(z) \prod_{k=1}^{s_\tau} \left\{\omega^{(l_{ki})}\right\}^{\nu_{ki}} = 0\,, \quad \alpha = \overline{1,\, h},$$

where $\lambda_\alpha$ and $h$ are defined from the condition (7.2).

For example for the equation

$$\sum_{i=0}^{H} A_i(z) \prod_{k=1}^{s_i} \left\{\omega^{(l_{ki})}\right\}^{\nu_{ki}} = \sum_{l=0}^{n} B_l(z) \left\{\omega^{(3)}\right\}^{\nu_l} \tag{7.3}$$

for $0 \le \nu_0 < \nu_1 < \ldots \nu_n,\ \nu_n \ge 2$, in accordance with theorem **6.6** and **6.7**, the famility (7.1) we will consider in the form

$$\omega = \varepsilon_t\, \mathcal{J}^3 \mathcal{G}(z), \qquad t = \overline{1,\, \delta}, \tag{7.4}$$

where $\delta = \nu_j - \nu_r$.

$$\mathcal{G}(z) \left[ \left(-B_r(z)/B_j(z)\right)^{1/\delta} \right]. \tag{7.5}$$

We'll define the polynomial $Q(z)$ from the correlation

$$B_r(z) = -B_j(z)\mathcal{G}^\delta(z) - Q(z)\,. \tag{7.6}$$

With the respect to (7.2) assume that the dimensions of the terms, situated in the left part of the equality (7.3), are connected by the correlation

$$\kappa_i = \kappa_0 + \theta_i\delta, \quad i = \overline{0, H}$$

where $\theta_i$ are integer numbers (in the investigations of D. Dimitrovski and P. Lazov [25] and L.G. Oreshchenko [37] for partial forms of equation (7.3) it is neccasary more strong condition $\theta_0 = \theta_1 = \ldots = \theta_H = 0$).

Then following statements take place.

**7.1.1.** For all polynomial of the famility (7.4) to be the solutions of equation (7.3) it is sufficient and for the limitations

$$\kappa_0 = \nu_{\tau_\lambda} + \xi_{\tau_\lambda}\delta, \quad \kappa_0 = \nu_\rho + \theta_1\delta, \quad \nu_h - \nu_l \neq \theta_2\delta, \quad \delta = \nu_j - \nu_r\,,$$

$$\tau_\lambda \in \{0, 1, \ldots, n\}, \quad \tau_\lambda \neq r, \quad r_\lambda \neq j, \quad \lambda = \overline{1, \beta}, \quad 1 \leq \beta \leq n-2,$$

$$l = \overline{0, n}, \quad l \neq r, \quad l \neq j, \quad l \neq \tau_\lambda, \quad h = \overline{0, n}, \quad h \neq j, \quad h \neq l,$$

$$h \neq \tau_\lambda, \quad \rho = \overline{0, n}, \quad \rho \neq j, \quad \rho \neq \tau_\lambda,$$

where $\theta_1 \neq 0$, $\theta_2 \neq 0$, $\xi_{\tau_\lambda}$ are integer numbers, it is necessary too that the equation (7.3) has the form

$$\sum_{i=0}^{H} A_i(z) \prod_{k=1}^{s_i} \left\{\omega^{(l_{ki})}\right\}^{\nu_{ki}} = \sum_{\lambda=1}^{\beta} B_{\tau_\lambda}(z) \left\{\omega^{(3)}\right\}^{\nu_{\tau_\lambda}} + $$
$$+ B_r(z)\left\{\omega^{(3)}\right\}^{\nu_r} + B_j(z)\left\{\omega^{(3)}\right\}^{\nu_j}$$

the polynomials $B_r(z)$ and $B_j(z)$ are connected with the correlation

$$B_r(z) = -B_j(z)\mathcal{G}^\delta(z)\,, \tag{7.7}$$

and the polynomial $\mathcal{J}^3\mathcal{G}(z)$ is the solution of the equation

$$\sum_{i=0}^{H} A_i(z) \prod_{k=1}^{s_i} \left\{\omega^{(l_{ki})}\right\}^{\nu_{ki}} = \sum_{\lambda=1}^{\beta} B_{\tau_\lambda}(z) \left\{\omega^{(3)}\right\}^{\nu_{\tau_\lambda}}\,.$$

**7.1.2.** For all polynomials of the famility (7.4) to be the solutions of the equation (7.3) it is sufficient and for the limitations

$$\kappa_0 = \nu_r + \xi_r\delta, \quad \kappa_0 \neq \nu_l + \theta_1\delta, \quad \nu_h - \nu_l \neq \theta_2\delta, \quad \delta = \nu_j - \nu_r.$$

$$l = \overline{0, n} \quad l \neq r, \quad l \neq j, \quad h = \overline{0, n}, \quad h \neq j, \quad h \neq l\,,$$

where $\xi_r$, $\theta_1 \neq 0$, $\theta_2 \neq 0$ are integer numbers, it is necessary too that the equation (7.3) has the form

$$\sum_{i=0}^{H} A_i(z) \prod_{k=1}^{s_i} \left\{\omega^{(l_{ki})}\right\}^{\nu_{ki}} = B_r(z) \left\{\omega^{(3)}\right\}^{\nu_r} + B_j(z) \left\{\omega^{(3)}\right\}^{\nu_j}, \tag{7.8}$$

where the polynomials $B_r(z)$ and $B_j(z)$ are connected with the correlation (7.6) and the polynomial $\mathcal{J}^3\mathcal{G}(z)$ is the solution of the equaion

$$\sum_{i=0}^{H} A_i(z) \prod_{k=1}^{s_i} \left\{\omega^{(l_{ki})}\right\}^{\nu_{ki}} = -Q(z) \left\{\omega^{(3)}\right\}^{\nu_r}.$$

**7.1.3.** For all polynomials of the famility (7.4) to be the solutions of the equation (7.3) it is sufficient and for the limitations

$$\kappa_0 = \nu_l + \xi_l\delta, \quad \nu_\rho - \nu_h \neq \theta\delta \quad l = \overline{0, n}, \quad h = \overline{0, n}, = \quad h \neq r, \quad h \neq j,$$

$$\rho = \overline{0, n}, \quad \rho \neq j, \quad \rho \neq h, \quad \delta = \nu_j - \nu_r ,$$

where $\xi_l$ and $\theta \neq 0$ are integer numbers, it is necessary too that this equation have the form (7.8), where the polynomials $B_r(z)$ and $B_j(z)$ are connected with the correlation (7.7), and the polynomial $\mathcal{J}^3\mathcal{G}(z)$ is the solution of the equation

$$\sum_{i=0}^{H} A_i(z) \prod_{k=1}^{s_i} \left\{\omega^{(l_{ki})}\right\}^{\nu_{ki}} = 0$$

Consider the famility of polynomials

$$\omega = \varepsilon_t\mathcal{G}(z) + D(z), \qquad t = \overline{1, \delta}, \tag{7.9}$$

of more general form which will be the solution of equation (1) then and only then when the system of the identities

$$\sum_{i=0}^{N} B_i(z) \sum_{j_1=0}^{\nu_{1_i}} \sum_{j_2=0}^{\nu_{2_i}} \cdots \sum_{j_{s_i}=0}^{\nu_{s_{i_i}}} \varepsilon_t^{\sum_{k=1}^{s_1} j_k} \prod_{k=1}^{s_i} \binom{\nu_{ki}}{j_k} * \tag{7.10}$$

$$* \left\{D^{(l_{ki})}(z)\right\}^{\nu_{ki}-j_k} \left\{\mathcal{G}^{(l_{ki})}(z)\right\}^{j_k} \equiv 0, \quad l = \overline{1, \delta},$$

is consistent.

Regroup the identities (7.10) taking into consideration thet the terms of each of these identities connected by one of logical possibilities

$$\sum_{k=1}^{s_i} j_k = \theta\delta, \; \sum_{k=1}^{s_i} j_k = 1 + \theta\delta, \ldots, \sum_{k=1}^{s_i} j_k = (\delta - 1) + \theta\delta, \tag{7.11}$$

where $\theta$ is integer non–negative number. Then the identities (7.10) will have the form

$$\sum_{\lambda=0}^{\delta-1} \varepsilon_t^{\lambda} \sum_{i=0}^{N} B_i(z) \sum_{\tau_1=0}^{\gamma_1(\lambda)} \sum_{\tau_2=0}^{\gamma_2(\lambda)} \cdots \sum_{\tau_{s_i}=0}^{\gamma_{s_i}(\lambda)} \prod_{k=1}^{s_i} \binom{\nu_{ki}}{{}_\lambda\chi_{k\tau_k}} * \tag{7.12}$$
$$* \left\{ D^{(l_{ki})}(z) \right\}^{\nu_{ki} - {}_\lambda\chi_{k\tau_k}} \left\{ \mathcal{G}^{(l_{ki})}(z) \right\}^{{}_\lambda\chi_{k\tau_k}} \equiv 0, \ \ \lambda = \overline{1, \delta}\,,$$

where ${}_\lambda\chi_{k\tau_k}$ $(k = \overline{1, s_i},\ \lambda = \overline{0, \delta - 1},\ \tau_k = \overline{0, \gamma_k(\lambda)})$ are meant those $j_k \in \{0, 1, \ldots, \nu_{ki}\}$, for which $\sum\limits_{k=1}^{s_i} j_k = \lambda + \theta\delta$, the numbers $\gamma_k(\lambda) \in \{0, 1, \ldots, \nu_{ki}\}$ and are defined in dependence of the grouping on (7.11).

*Theorem* **7.2**. For all polynomials of the famility (7.9) to be the solutions of algebraic differential equation (1) it is necessary and sufficient the fulfilment of the famility of the identities

$$\sum_{i=1}^{N} B_i(z) \sum_{\tau_1=0}^{\gamma_1(\lambda)} \sum_{\tau_2=0}^{\gamma_2(\lambda)} \cdots \sum_{\tau_{s_i}=0}^{\gamma_{s_i}(\lambda)} \prod_{k=1}^{s_t} \binom{\nu_{ki}}{{}_\lambda\chi_{k\tau_k}} *$$
$$* \left\{ D^{(l_{ki})}(z) \right\}^{\nu_{ki} - {}_\lambda\chi_{k\tau_k}} \left\{ \mathcal{G}^{l_{ki}}(z) \right\}^{{}_\lambda\chi_{k\tau_k}} \equiv 0, \ \ \lambda = \overline{0, \delta - 1}\,,$$

where the numbers $\gamma_k(\lambda)$ and ${}_\lambda\chi_{k\tau_k}$ are defined in dependence of the grouping on $\lambda$ on the base of (7.11), as this is pointed in the identities (7.12).

Theorem **7.2** have general character but the representations are higly cumbersome. Therefore consider partial forms of equation (1).

For algebraic differential equation

$$\sum_{i=1}^{H} A_i(z)\omega^{(s_t)} + \sum_{l=0}^{n} B_l(z) \left\{ \omega^{(\mathfrak{z})} \right\}^{\nu_l} = 0 \tag{7.13}$$

for $0 \le s_0 < s_1 < \ldots < s_H$, $s_t \ne \mathfrak{z}$, $i = \overline{0, H}$, in accordance whith theorem **6.7**, the famility (7.9) we will find in the form

$$\omega = \varepsilon_t \mathcal{J}^{\mathfrak{z}} \mathcal{G}(z) + D(z), \qquad t = \overline{1, \delta}\,, \tag{7.14}$$

where $\delta = \nu_j - \nu_r$, the polynomial is defined by the formula (7.5), $D(z)$ is the polynomial of degree $\deg D(z) = d < \mathfrak{z}$, and the equality (7.6) takes a place.

In the set $\{0, 1, \ldots, \delta - 1\}$ there is always the number $\rho$ such that $\nu_r = \rho + \theta\delta$, where $\theta$ is the integer number. In accordance with the integer numbers $0 \le q_0 < q_1 < \ldots < q_h \le \delta - 1$ the exponents of degree $\nu_l$, $l = \overline{0, n}$, $l \ne r$, $l \ne j$, we'll situats so that $\nu_{l_\eta + \tau} = q_\eta + \xi_{l_\eta} + \tau^\delta$,

$$\tau = \overline{0, \xi_n}, \quad \xi_n \ge 0, \quad \eta = \overline{0, h}, \quad \sum_{\eta=0}^{h} (\xi_\eta + 1) = n - 1\,,$$

where $\xi_{l_\eta+\tau}$ are the integer numbers, i.e. the set

$$\{\nu_{l_0}, \nu_{l_0+1}, \ldots, \nu_{l_0+\xi_0}, \nu_{l_1}, \nu_{l_1+1}, \ldots, \nu_{l_1+\xi_1}, \ldots, \nu_{l_h}, \nu_{l_h+1}, \ldots, \nu_{l_h+\xi_h}\}$$

is rearrangement of the set $\{\nu_0, \nu_1, \ldots, \nu_n\} \setminus \{\nu_r, \nu_j\}$, such that, if

$$\nu_\lambda = q_k + \xi_\lambda \delta$$

then either

$$\nu_{\lambda+1} = q_k + \xi_{k+1}\,\delta \text{ or}$$

$$\nu_{\lambda+1} = q_{k+1} + \xi_{k+1}\,\delta$$

for $q_k < q_{k+1}$ for any $\lambda = \{0, 1, \ldots, n\} \setminus \{r, j\}$.

Then the following statements are correct.

**7.2.1.** For all polynomials of famility (7.14) to be the solutions of equation (7.13), it is sufficient and for $\rho = 0$ it is necessary too for polynomials $B_r(z)$ and $B_j(z)$ to be connected by the correlations (7.6) and the polynomial $\mathcal{J}^3\mathcal{G}(z)$ to be the solution of the equation's system

$$\sum_{i=0}^{H} A_i(z)\omega^{(s_i)} + \sum_{k=0}^{1} \delta^1_{qk} \sum_{\tau=0}^{\xi_k} B_{l_k+\tau}(z) \left\{\omega^{(3)}\right\}^{\nu_{l_k+\tau}} = 0\,,$$

$$\left(i - \delta^0_{q_\eta}\right)\left(1 - \delta^1_{q_\eta}\right) \sum_{\tau=0}^{\xi_\eta} B_{l_\eta+\tau}(z) \left\{\omega^{(3)}\right\}^{\nu_{l_\eta+\tau}} = 0, \quad \eta = \overline{o, h}\,.$$

and for the polynomials $D(z)$ to satisfy with the equation

$$\sum_{i=0}^{H} A_i(z)\omega^{(s_i)} + \delta^0_{q_0} \sum_{\tau=0}^{\xi_0} B_{l_0+\tau}(z)\,\mathcal{G}^{\nu_{l_0+\tau}}(z) - Q(z)\mathcal{G}^{\nu_r}(z) = 0\,,$$

where $\delta^i_j$ is Kronecker's symbol.

**7.2.2.** For all polynomials of famility (7.14) to be the solutions of equation (7.13), it is sufficient and for $\rho = 1$ it is necessary too for the polynomials $B_r(z)$ and $B_j(z)$ to be connected by the correlations (7.6) for the plynomial $\mathcal{J}^3\mathcal{G}(z)$ to be the solution of the equation's system

$$\sum_{i=0}^{H} A_i(z)\omega^{(s_i)} + \sum_{k=0}^{1} \delta^1_{q_k} \sum_{\tau=0}^{\xi_k} B_{l_k+\tau}(z) \left\{\omega^{(3)}\right\}^{\nu_{l_k+\tau}} -$$

$$-Q(z)\left\{\omega^{(3)}\right\}^{\nu_r} \mathcal{G}^{\nu_r}(z) = 0,$$

$$\left(1 - \delta^0_{q_\eta}\right)\left(1 - \delta^1_{q_\eta}\right) \sum_{\tau=0}^{\xi_\eta} B_{l_\eta+\tau}(z) \left\{\omega^{(3)}\right\}^{\nu_{l_\eta+\tau}} = 0, \quad \eta = \overline{0, h}\,,$$

and for the polynomials $D(z)$ to satisfy with the equation

$$\sum_{i=0}^{H} A_i(z)\omega^{(s_i)} + \delta_{q_0}^0 \sum_{\tau=0}^{\xi_0} B_{l_0+\tau}(z)\mathcal{G}^{\nu_{l_0+\tau}}(z) = 0 . \qquad (7.15)$$

**7.2.3.** For all polynomials of famility (7.14) to be the solutions of equation (7.13), it is sufficient, and for $\rho = q_\lambda$, $\rho > 1$, $\lambda \in \{0, 1, \dots, h\}$, it is necessary too for the polynomials $B_r(z)$ and $B_j(z)$ to be conected by the correlation (7.6), for the polynomial $\mathcal{J}^3\mathcal{G}(z)$ to be the solution of the equation's system

$$\sum_{i=0}^{H} A_i(z)\omega^{(s_i)} + \sum_{k=0}^{1} \delta_{q_k}^1 \sum_{\tau=0}^{\xi_k} B_{l_k+\tau}(z)\left\{\omega^{(3)}\right\}^{\nu_{l_k+\tau}} = 0 ,$$

$$\sum_{\tau=0}^{\xi_\lambda} B_{l_\lambda+\tau}(z)\left\{\omega^{(3)}\right\}^{\nu_{l_\lambda+\tau}} - Q(z)\left\{\omega^{(3)}\right\}^{\nu_r} = 0$$

$$\left(1-\delta_{q_\eta}^0\right)\left(1-\delta_{q_\eta}^1\right)\sum_{\tau=0}^{\xi_\eta} B_{l_\eta+\tau}(z)\left\{\omega^{(3)}\right\}^{\nu_{l_\eta+\tau}} = 0, \quad \eta = \overline{0, h} .$$

and for the polynomials $D(z)$ to satisfy with the equation (7.15)

**7.2.4.** For all polynomials of famility (7.14) to be the solution of equation (7.13), it is sufficient, and for $\rho > 1$, $\rho \neq q_\eta$, $\eta = \overline{0, h}$, it is necessary too for the polynomials $B_r(z)$ and $B_j(z)$ to be connected by the correlation (7.7), for the polynomial $\mathcal{J}^3\mathcal{G}(z)$ to be the solution of the system

$$\sum_{i=0}^{H} A_i(z)\omega^{(s_i)} + \sum_{k=0}^{1} \delta_{q_k}^1 \sum_{\tau=0}^{\xi_k} B_{l_k+\tau}(z)\left\{\omega^{(3)}\right\}^{\nu_{l_k+\tau}} = 0 ,$$

$$\left(1-\delta_{q_0}^0\right)\left(1-\delta_{q_1}^1\right)\sum_{\tau=0}^{\xi_\eta} B_{l_\eta+\tau}(z)\left\{\omega^{(3)}\right\}^{\nu_{l_\eta+\tau}} = 0, \quad \eta = \overline{0, h} ,$$

and for the polynomials $D(z)$ to satisfy with the equation (7.15).

## References *

[1] Gorbuzov, V. N.: *Polynomial solutions of algebraic ordinary differential equations* I, Bulletin de la societé des mathem. et des informar. de Macedonie, Skopje, **17**, 5–20, 1993

[2] Gorbuzov, V. N.: *Polynomial solutions of algebraic ordinary differential equations* II, Bulletin de la societé des mathem. et des informar. de Macedonie, Skopje, **18**, 5–16, 1994

* Note of the editor: The reference in this paper is the sameas in the paper [1]

# ПОЛИНОМНИ РЕШЕНИЈА НА АЛГЕБАРСКИ ОБИЧНИ ДИФЕРЕНЦИЈАЛНИ РАВЕНКИ. III

Б. Н. Горбузов

**Р е з и м е**

Се посматра градбата на полиномните решенија на алгебарските диференцијални равенки.

Гродненский госуниверситет

Кафедра математического анализа

ул. Ожешко 22

230023, Гродно

Република Беларусь