Математички Билтен
20 (XX)
1996 (5–16)
Скопје, Македонија

ISSN 0351–336X

FREE OBJECTS IN SOME VARIETIES OF GROUPOIDS

Ǵ. Čupona* and N. Celakoski **

Abstract

We give a canonical description of free objects in the variety $\mathcal{V}_{m,n}$ of groupoids which satisfy the law $x^m y^n = z_1 z_2 \dots z_m$, where $z_i = x$ if i is odd, $z_j = y$ if j is even, and m, n are integers such that $m > n \geq 2$. We also consider a derived quasivariety $\mathcal{V}^{\square}_{m,n}$ of groupoids in which only trivial identities hold.

0. Introduction

A *groupoid* is an algebra $\mathbf{G} = (G, \cdot)$ with one binary operation $(x, y) \mapsto x \cdot y$. As usual, the symbol of the operation and some brackets will be omited. Namely, if $a, a_1, a_2, \dots, a_k, a_{k+1} \in G$, then:

$$a^1 = a, \qquad a^{k+1} = a^k a, \tag{0.1}$$

$$a_1 a_2 \cdots a_k a_{k+1} = (a_1 a_2 \cdots a_k) a_{k+1}. \tag{0.2}$$

If k is a positive integer and $a, b, c \in G$, $x_i = a$, $x_j = b$ for $1 \leq i, j \leq k$, where i is odd and j is even, then:

$$\begin{gathered} \underline{abk} = x_1 x_2 \cdots x_k, \qquad \underline{ab1} = a, \\ c\,\underline{abk} = c x_1 \cdots x_k, \quad c\,\underline{ab1} = ca, \quad c\,\underline{ab0} = c. \end{gathered} \tag{0.3}$$

Note that if $a, b, a_\nu, b_\nu \in G$, $k > 0$, $k_1, k_2, \dots \geq 0$, then

$$\underline{abk}\,\underline{a_1 b_1 k_1} \dots \underline{a_j b_j k_j} \tag{0.4}$$

is an element of G which is defined by:

$$\underline{abk}\ \underline{a_1b_10} = \underline{abk}, \quad \underline{abk}\,\underline{a_1b_11} = (\underline{abk})\cdot a_1,$$
$$\underline{abk}\ \underline{a_1b_12} = \big((\underline{abk})a_1\big)b_1, \ldots$$

It should be pointed out that, in (0.4), $\underline{abk}$ is an element of G, and each of the triples $\underline{a_ib_ik_i}$ takes part as a sequence of elements where the multiplication is "from left to the right" according to the definition (0.2). For example: $\underline{ab3}$, $\underline{cd2} \in G$, but $\underline{ab3}\ \underline{cd2} \neq (\underline{ab3})\cdot(\underline{cd2})$. Namely,

$$\underline{ab3}\ \underline{cd2} = \Big(\big((ab)a\big)c\Big)d = abacd,$$
$$(\underline{ab3})\cdot(\underline{cd2}) = \big((ab)a\big)(cd).$$

Recall that $\mathcal{V}_{m,n}$ is the variety of groupoids which satisfy the identity

$$x^m\, y^n = \underline{xym}, \tag{0.5}$$

where m, n are positive integers. Further on we will assume that $m > n \geq 2$, if it is not stated otherwise.

For every $p \geq 0$, we define transformations $x \mapsto x^{<p>}$ and $x \mapsto x^{(p)}$ of G in the following way:

$$x^{<0>} = x^{(0)} = x, \quad x^{<p+1>} = \left(x^{<p>}\right)^m, \quad x^{(p+1)} = \left(x^{(p)}\right)^n. \tag{0.6}$$

Clearly:

$$\left(x^{<p>}\right)^{<q>} = x^{<p+q>}, \quad \left(x^{(p)}\right)^{(q)} = x^{(p+q)}, \tag{0.7}$$

for all p, $q \geq 0$.

Let $\mathbf{Q} = (Q, \circ)$ and $\mathbf{G} = (G, \cdot)$ be groupoids such that $Q \subseteq G$. Q is said to be an (m, n)*–subgroupoid* of $\mathbf{G}$ iff $a \circ b = a^m b^n$ for all $a, b \in Q$. The class of groupoids which are (m, n)–subgroupoids of groupoids in $\mathcal{V}_{m,n}$ will be denoted by $\mathcal{V}^{\square}_{m,n}$. So $\mathcal{V}^{\square}_{m,n}$ is derived from $\mathcal{V}_{m,n}$. ([3], III.7.)

A free groupoid (in the variety of all groupoids) with a given basis B will be denoted by $\mathbf{F} = (F, \cdot)$.

We denote by $R_{m,n}$ the least subset of F such that $B \subset F$ and

$$xy \in R_{m,n} \Leftrightarrow [x, y \in R_{m,n} \text{ and } (\forall \alpha, \beta \in F)(x \neq \alpha^m \text{ or } y \neq \beta^n)]. \tag{0.8}$$

(Further on we will write R instead of $R_{m,n}$.)

Below we define a mapping $*: R^2 \to F$.

Let x, $y \in R$ be such that $xy \in R$ and $[y]_n = r$ 1), $y = z^{(r)}$. Then, we define $x * y$, $x^{<1>} * y^{(1)}$ and $x^{<p+1>} * y^{(p+1)}$, where $p \geq 1$, as follows.

$$x * y = xy. \tag{0.9}$$

$$x^{<1>} * y^{(1)} = \begin{cases} y^3, & \text{if } x = y^2,\ n = 2,\ m = 3 \\ z^{n+2}\ \underline{z^{(1)}\, z\, n-3} \ldots \underline{z^{(r+1)} z^{(r)}\ n-3}, & \\ & \text{if } x = y^n,\ n \geq 3,\ m = n+1 \\ \underline{xym}, & \text{if } x \neq y^n \text{ or } m > n+1\,. \end{cases} \tag{0.10}$$

$$x^{<p+1>} * y^{(p+1)} = \left(x^{<1>} * y^{(1)}\right) \underline{x^{<1>} y^{(1)} m-2} \cdots \underline{x^{<p>} y^{(p)} m-2} \tag{0.11}$$

The following theorems are the main results in the paper.

THEOREM 1. *$u{*}v \in R$, for all $u, v \in R$ and the groupoid $\mathbf{R} = (R, *)$ is free in $\mathcal{V}_{m,n}$ with the (unique*) basis B.

THEOREM 2. *$\mathcal{V}^{\square}_{m,n}$ is a proper quasi–variety of groupoids, and only trivial identities hold in $\mathcal{V}^{\square}_{m,n}$.*

REMARK. (m, n)–subgroupoids are special kinds of t–subgroupoids, where $t = t(x, y)$ is a groupoid term in which two variables x, y appear. ($\mathbf{Q} = (Q, \circ)$ is a t–subgroupoid of a groupoid $\mathbf{G} = (G, \cdot)$ iff $Q \subseteq G$ and

$$a \circ b = t_{\mathbf{G}}(a, b)\,, \tag{0.12}$$

for all $a, b \in Q$; the right–hand side of (0.12) is the value of the term $t(x, y)$ in $\mathbf{G}$ for $x = a$, $y = b$.) If $\mathcal{V}$ is a variety of groupoids, then the class of t–subgroupoids in $\mathcal{V}$ will be denoted by $\mathcal{V}^t$.

The paper [7] consider a question which can be "translated" in the language of groupoids in the following way: "Is the condition «Only trivial identities hold in $\mathcal{V}^t$» sufficient for the class $\mathcal{V}^t$ to coincide with the variety of all groupoids?" The answer (which follows by *Th. 2*) is negative. The question: whether the same is true for generalized subalgebras of algebras of any type Ω, remains open.

Th. 1, *Th.* 2 are proved in §1, §2 respectively. The obtained canonical description of free groupoids in $\mathcal{V}$ (in *Th.* 1) is due to the fact that the rewriting system on F induced by elementary transformations

---

1) $[y]_n$ is the largest non–negative integer r such that $y = z^{(r)}$ for some $z \in F_n$. (See also below, after (1.4.2).)

$u^m v^n \to \underline{uvm}$ is a terminating Church–Rose system. This conclusion does not hold in the case $2 \le m \le n$ or $m > n = 1$, which is shown in §3.

1. A canonical description of free groupoids in $\mathcal{V}_{m,n}$

In this section we will prove *Th.* 1 in the case $m > n \ge 2$; first we state some properties of **F**.

The following two properties are characteristic for a free groupoid **F** with the basis B ([1], I.1).

a) $ab = cd \Rightarrow a = c,\ b = d$.

(Any groupoid with this property is said to be *injective.*)

b) *B is the set of primes in* **F** *and it generates* **F**.

(An element $c \in G$ is *prime* in a groupoid $\mathbf{G} = (G, \cdot)$ iff $c \ne xy$, for all $x, y \in G$.)

The *norm* in **F** is the homomorphism $x \mapsto |x|$ from **F** into the additive groupoid of positive integers which is an extension od the mapping $B \to \{1\}$. Thus:

$$|b| = 1\,, \quad |uv| = |u| + |v|\,, \tag{1.1}$$

for $b \in B$, u, $v \in F$.

The statements below are direct consequences of (1.1) and the injectivity of **F**. Here: x_ν, y_ν, x, y, $\alpha \in F$, $i, j, k \ge 1$, $p, q, r \ge 0$.

$$|x^i| = i\,|x|\,, \quad |x^{<p>}| = m^p\,|x|\,, \quad |x^{(p)}| = n^p\,|x|\,. \tag{1.2}$$

$$x^i = y^j \Rightarrow x = y\,, \quad i = j\,; \tag{1.3.1}$$

$$x_1 x_2 \ldots x_i y_1 \ldots y_j = z z_1 \ldots z_j \Rightarrow \Rightarrow z = x_1 \ldots x_i,\ z_1 = y_1, \ldots, z_j = y_j\,; \tag{1.3.2}$$

$$x^{<p>} = y^{<p+q>} \Rightarrow x = y^{<q>}, \qquad x^{(p)} = y^{(p+q)} \Rightarrow x = y^{(q)}. \tag{1.3.3}$$

$$2 \le k \le m\,,\ x_1 \ne x_i \text{ for some } i > 1 \Rightarrow x_1 x_2 \ldots x_k \ne \alpha^m\,; \tag{1.4.1}$$

$$2 \le k < m\,,\ y_i \ne y_j \text{ for some } i \ne j \Rightarrow x_1 \ldots x_p y_1 \ldots y_k \ne \alpha^m\,; \tag{1.4.2}$$

According to (1.2), for any $u \in F$, there exists the largest non-negative integer k such that $u = x^{<k>}$, for some $x \in F$. This integer k will be denoted by $[u]_m$. One defines $[u]_n$ in the same way. Next, $[u, v]$ is defined by:

$$[u,\ v] = \min\{[u]_m, [u]_n\}\,. \tag{1.5}$$

By (1.5), it follows:

$$\left[u^{<p>},\ v^{(p)}\right] = p + [u,\ v]\,. \tag{1.6}$$

The definition (0.8) can obtain now the following form:

$$uv \in R_{m,n} \Leftrightarrow (u,\, v \in R_{m,n},\ \ [u,\, v] = 0)\,. \tag{1.7}$$

(As above, we will write R instead of $R_{m,n}$.)

The following properties are also consequences of (1.7) and (1.2)–(1.6).

$$1 \leq k \leq m,\ x \in R \Rightarrow x^k \in R,\ \ x^{<p>},\ x^{(p)} \in R\,. \tag{1.8}$$

$$x^{<p>} \in R \text{ or } x^{(p)} \in R \Rightarrow x \in R\,. \tag{1.9}$$

$$x \in R \Rightarrow \left(x^{m+1} \in R \Leftrightarrow [x]_n = 0\right)\,. \tag{1.10}$$

$$p \geq 1,\ \ x,\, y \in R \Rightarrow xy^{<p>} \in R\,. \tag{1.11}$$

$$p \geq 1,\ \ x, y \in R \Rightarrow \left(xy^{(p)} \in R \Leftrightarrow [x]_m = 0\right)\,. \tag{1.12}$$

$$xy \in R \Rightarrow \left(xyx \in R \Leftrightarrow (x \neq y^n \text{ or } m > n+1)\right)\,. \tag{1.13}$$

Assume now that $u * v \in F$ is defined by (0.9), (0.10) and (0.11), where $u,\, v \in R$ are such that $[u,\, v] = p$, $u = x^{<p>}$, $y = y^{(p)}$, $[y]_n = r$, $y = z^{(r)}$. We have to show that $u * v \in R$.

If $p = 0$, then $u * v = uv \in R$, and thus we can assume that $p \geq 1$. Consider first the case $x \neq y^n$ or $m > n+1$. Then $x^{<1>} * y^{(1)} = \underline{xym}$. By (1.13) we have $xyx = \underline{xy3} \in R$, and thus we can assume that $m \geq 4$. Then from (1.4.1) it follows: $xyxy \in R$. In the same way one can obtain: $x^{<1>} * y^{(1)} = \underline{xym} \in R$. Assume now that $p \geq 1$. Then:

$$x^{<p+1>} * y^{(p+1)} = \underline{xym}\ \underline{x^{<1>}y^{(1)}m-2} \ldots \underline{x^{<p>}y^{(p)}m-2}\,. \tag{1.14}$$

We will consider only the case $m = 4$ (and $n = 2$ or $n = 3$). Then:

$$x^{<p+1>} * y^{(p+1)} = xyxy\ \ x^{<1>}y^{(1)} \ldots x^{<p>}y^{(p)}\,. \tag{1.14$'$}$$

From (1.11) we obtain $xyxyx^{<1>} \in R$, and then (1.4.2) implies: $xyxyx^{<1>}y^{(1)} \in R$. Continuing in this way we would get $x^{<p+1>} * y^{(p+1)} \in R$.

It remains the case $x = y^n$, $m = n+1$.

If $n = 2$, then $x^{<1>} * y^{(1)} = y^3 \in R$, and therefore $x^{<p+1>} * y^{(p+1)} = y^3x^{<1>} \ldots x^{<p>} \in R$, by (1.11). Thus we can assume that $n \geq 3$. In the case $n = 3$, we have $x^{<1>} * y^{(1)} = z^5 \in R$, by (1.10), and then (in the same way as in the case $m = 4, x \neq y^2$) one can show that $x^{<p+1)} * y^{(p+1)} \in R$, in the case $m = n+1 = 4$, $x = y^3$, as well. It remains the case $n+1 = m \geq 5$, $y = x^n$. Then, we obtain $x^{<1>} * y^{(1)} \in R$, by applications of (0.10), (1.10) and (1.4.2). Finally, in the same way as in the first considered case ($x \neq y^n$ or $m > n+1$) one can obtain that $x^{<p+1>} * y^{(p+1)} \in R$.

Thus we have the following:

PROPOSITION 1.1. $\mathbf{R} = (R, *)$ *is a groupoid* . $\diamondsuit$

Below we will show that $(R, *) \in \mathcal{V}_{m,n}$. First, denote by u_*^k ($u \in R$, $k \geq 1$) the corresponding k–th power of u in $\mathbf{R}$, i.e.

$$u_*^1 = u\,, \qquad u_*^{k+1} = u_*^k * u\,.$$

By (0.9) and (1.8): $k \leq m \Rightarrow u_*^k = u^k$, and thus:

$$u_*^m = u^m, \quad u_*^n = u^n, \quad u_*^{<p>} = u^{<p>}, \quad u_*^{(p)} = u^{(p)}, \tag{1.15}$$

for all $u \in R$, $p \geq 0$. This implies:

$$u_*^m * v_*^n = u^m * v^n = x^{<p+1>} * y^{(p+1)}, \tag{1.16}$$

where $u, v \in R$, $[u, v] = p$, $u = x^{<p>}$, $v = y^{(p)}$.

If $u, v \in R$, then $\underline{*uvm}$ will be an abbreviation for the product $z_1 * z_2 * \cdots * z_m$, where $z_i = u$ when i is odd, and $z_j = v$ when j is even. (Note that $\underline{uvm} \in F$, $\underline{*uvm} \in R$ and it is possible $\underline{*uvm} \neq \underline{uvm}$.)

From (0.9), (0.10) and (0.11) we obtain

$$\begin{aligned} \underline{*xym} &= x^{<1>} * y^{(1)}, \\ \underline{*x^{<p>}y^{(p)}m} &= \left(x^{<1>} * y^{(1)}\right) \underline{x^{<1>}y^{(1)}m-2} \cdots \underline{x^{<p>}y^{(p)}m-2}\,. \end{aligned} \tag{1.17}$$

(For example, if ($m = 4$, $n = 2$) or ($m = 4$, $n = 3$, $x \neq y^3$), then:

$$\underline{*xy4} = \big((x * y) * x\big) * y = xyxy = x^{<1>} * y^{(1)}\,.$$

If $p \geq 1$, then

$$\begin{aligned} \underline{*x^{<p>}y^{(p)}4} &= \big((x^{<p>} * y^{(p)}) * x^{<p>}\big) * y^{(p)} \\ &= xyxyx^{<1>}y^{<1>} \ldots x^{<p>}y^{(p)} \\ &= (x^{<1>} * y^{(1)})\, x^{<1>}y^{(1)} \ldots x^{<p>}y^{(p)}\,. \end{aligned}$$

In the case $n+1 = m = 4$, $x = y^3$, $[y]_n = r$, $y = z^{(r)}$, we have:

$$\underline{*xy4} = \big((x * y) * x\big) * y = z^5 = x^{<1>} * y^{(1)},$$

$$\underline{*\,x^{<p>}y^{(p)}\,4} = ((x^{<p>} * y^{(p)}) * x^{<p>}) * y^{(p)}$$
$$= z^5 x^{<1>} y^{(1)} \dots x^{<p>} y^{(p)}$$
$$= (x^{<1>} * y^{(1)})\, x^{<1>} y^{(1)} \dots x^{<p>} y^{(p)} .)$$

So, the following equation holds:

$$u^m * v^n = \underline{*\,uvm}\,, \tag{1.18}$$

and therefore we obtain:

PROPOSITION 1.2. $R \in \mathcal{V}_{m,n}$. ◇

The following statement "inspired" the definition of R and $*$, and it will be used in the proof of *Pr.* 1.4, as well.

PROPOSITION 1.3. *If* $\mathbf{G} = (G, \cdot) \in \mathcal{V}_{m,n}$, *then the following implications hold*:

a) $p \geq 1 \Rightarrow x^{<p+1>}y^{(p+1)} = \underline{xym}\ \underline{x^{<1>}y^{(1)}m-2} \dots \underline{x^{<p>}y^{(p)}m-2}.$

b) $m = 3 = n+1 \Rightarrow y^3y^2 = y^3.$

c) $m = 4 = n+1,\ r \geq 0 \Rightarrow \left(z^{(r+1)}\right)^4 z^{(r+1)} = z^5.$

d) $m = n+1 \geq 5,\ r \geq 0 \Rightarrow$

$\left(z^{(r+1)}\right)^m z^{(r+1)} = z^{m+1}\,\underline{z^{(1)}zm-2} \dots \underline{z^{(r+1)}z^{(r)}m-2}.$ ◇

PROPOSITION 1.4. *If* $\mathbf{G} = (G, \cdot) \in \mathcal{V}_{m,n}$ *and* $\varphi \colon F \to G$ *is a homomorphism from* $\mathbf{F}$ into $\mathbf{G}$ *then the restriction* ψ of φ on R *is a homomorphism from* $\mathbf{R}$ into $\mathbf{G}$.

Proof. By using *Pr.* 1.3 and the definition of $*$. ◇

As a consequence we obtain *Th.* 1, i.e. the following

PROPOSITION 1.5. $\mathbf{R}$ *is free in* $\mathcal{V}_{m,n}$ *with the (unique) basis* B.

Proof. First, by the definition of R and $*$, B is the set of primes in $\mathbf{R}$ and B generates $\mathbf{R}$. Let $\mathbf{G} = (G, \cdot) \in \mathcal{V}_{m,n}$ and $\lambda \colon B \to G$ be a mapping. If φ is the homomorphism from $\mathbf{F}$ into $\mathbf{G}$ which extends λ, then by *Pr.* 1.4, the restriction ψ of φ on R is a homomorphism from $\mathbf{R}$ into $\mathbf{G}$. ◇

Below we show a variant of *Th.* 1 concerning the variety $\mathcal{V}_{1,n}$.

PROPOSITION 1.6. $\mathcal{V}_{1,1}$ *is the variety of left–zero groupoids.* ◇

PROPOSITION 1.7. *If* $m = 1,\ n \geq 2$, *then* $[u]_n \leq 1$ *for every* $u \in R(= R_{1,n})$. *If an operation* $*$ *is defined by*:

$$u * v = \begin{cases} uv, & \text{if } [v]_n = 0, \\ u, & \text{if } [v]_n = 1, \end{cases} \tag{1.19}$$

then $\mathbf{R} = (R, *)$ *is a groupoid which is free in* $\mathcal{V}_{1,n}$ *with the basis* B.

Proof. It is clear by (0.8) that $[v]_n \geq 2$ implies $v \notin R$. Therefore, by (1.19), we obtain:

$$v_*^n = \begin{cases} v^n, & \text{if} \quad [v]_n = 0, \\ v, & \text{if} \quad [v]_n = 1, \end{cases}$$

and so:

$$u * v_*^n = \begin{cases} u * v^n = u, & \text{if} \quad [v]_n = 0, \\ u * v = u, & \text{if} \quad [v]_n = 1. \end{cases}$$

Thus, $\mathbf{R} \in \mathcal{V}_{1,n}$. ◇

2. Some properties of the class $\mathcal{V}^{\square}_{m,n}$

Given a groupoid $\mathbf{H} = (H, \cdot)$, by $\mathbf{H}^{\square}$ will be denoted the groupoid $(H, \square)$ defined by

$$a \,\square\, b = a^m b^n \tag{2.1}$$

(The right–hand side of (2.1) has the usual meaning in $\mathbf{H}$.)

In *Pr.* 2.1–*Pr.* 2.5, m, n are (arbitrary) positive integers.

PROPOSITION 2.1. $\mathbf{G} \in \mathcal{V}^{\square}_{m,n}$ *iff there exists a groupoid* $\mathbf{H} \in \mathcal{V}_{m,n}$ *such that* $\mathbf{G}$ *is a subgroupoid of* $\mathbf{H}^{\square}$. ◇

Propositions 2.2 and 2.3 are special cases of more general results. (For example: [3], IV.5 and IV.6; [8], V.11.2.)

PROPOSITION 2.2. *Let* $\mathbf{G} = (G, \circ)$ *be a groupoid and* $\mathbf{R} = (R, *)$ *be a free groupoid in* $\mathcal{V}_{m,n}$ *with the basis* G. *Let* $\approx$ *be the least congruence on* $\mathbf{R}$ *with the property*

$$a \circ b = c \Rightarrow a^m * b^n \approx c. \tag{2.2}$$

Then: $\mathbf{G} \in \mathcal{V}^{\square}_{m,n}$ *iff the following condition is satisfied:*

$$(\forall\, a, b \in G)[a \approx b \Rightarrow a = b]. \quad ◇$$

PROPOSITION 2.3. $\mathcal{V}^{\square}_{m,n}$ *is a quasi–variety, i.e. there exists a system of axioms of* $\mathcal{V}^{\square}_{m,n}$ *each of which is a quasi–identity.*[3] ◇.

---

3) *Pr.* 2.2 is almost obvious and *Pr.* 2.3 is a corollary of it. Moreover, we can use *Pr.* 2.2 to obtain a convenient axiom system for $\mathcal{V}^{\square}_{m,n}$. Such a procedure is exposed in [2], where it is found an axiom system of quasi–identities for the quasi–variety of algebras $\mathbf{A} = (A, \Omega)$ which can be embedded in semigroups $\mathbf{S} = (S, \cdot)$ in such a way that $f(a_1, \ldots, a_n) = a_1 \ldots a_n$, for each n–ary operator $f \in \Omega_n$ $(n \geq 2)$.

PROPOSITION 2.4. *The quasi–identity*

$$x \circ x = y \circ y \Rightarrow x \circ z = y \circ z \tag{2.3}$$

is true in each groupoid $\mathbf{G} = (G, \circ) \in \mathcal{V}^{\square}_{m,n}$. ◇

PROPOSITION 2.5. $\mathcal{V}^{\square}_{m,n}$ *is a proper subclass of the class of groupoids.*

Proof. Let $\mathbf{G} = (\{a, b\}, \cdot)$ be a two–element groupoid such that $ba = b$, and $xy = a$ in every other case. Then (2.3) is not satisfied in $\mathbf{G}$. ◇

Below we will establish some properties of the groupoid $\mathbf{R}^{\square} = (R, \square)$, assuming that $m > n \geq 2$. First recall that

$$u \square v = u^m * v^n, \tag{2.1'}$$

for all $u, v \in R$.

In the *Pr.* 2.6–2.11 we assume that $m > n \geq 2$. They are corolaries of the definitions of $\mathbf{R}$ and $\mathbf{R}^{\square}$, and the injectivity of $\mathbf{F}$.

PROPOSITION 2.6. x^n *is a prime in* $R^{\square}$, *for each* $x \in \mathbf{R}$. ◇

PROPOSITION 2.7. *If* $(m, n) \notin \{(3,2), (4,3)\}$, *then* $\mathbf{R}^{\square}$ *is injective.* ◇

PROPOSITION 2.8. *Let* $u, v, \gamma, \delta \in R$ *and* $(u, v) \neq (\gamma, \delta)$.

1) *If* $(m, n) = (3,2)$, *then:*

$u \square v = \gamma \square \delta$ *iff* $\{(u, v), (\gamma, \delta)\} = \{(y^2, y), (y, y)\}$, *for some* $y \in R$.

2) *If* $(m, n) = (4, 3)$, *then:* $u \square v = \gamma \square \delta$ *iff*

$$\{(u, v), (\gamma, \delta)\} = \{(z^{(r+1)}, z^{(r)}), (z^{(s+1)}, z^{(s)})\},$$

for some $z \in R$ *and* $0 \leq r < s$. ◇

PROPOSITION 2.9. *The subgroupoid* $\mathbf{Q}$ *of* $\mathbf{R}^{\square}$ *generated by the basis* B *of* $\mathbf{R}$ *is injective.* [4)]

Proof. If $(m, n) \in \{(3, 2), (4, 4)\}$, the assertion is a corollary from *Pr.* 2.6 and *Pr.* 2.8; in the case $m > n + 1$ or $m \geq 5$ we can apply *Pr.* 2.7. ◇

PROPOSITION 2.10. *Only trivial identities hold in* $\mathcal{V}^{\square}_{m,n}$. ◇

Finally:

PROPOSITION 2.11. $\mathcal{V}^{\square}_{m,n}$ *is not a variety.*

---

4) We note (see, for example: [3], IV.4.4) that $\mathbf{Q}$ is free in $\mathcal{V}^{\square}_{m,n}$ with basis B.

Proof. If $\mathcal{V}^{\square}_{m,n}$ were a variety, then by *Pr.* 2.10, it would be defined by a trivial identity, for example $x = x$. This would imply that $\mathcal{V}^{\square}_{m,n}$ is the class of all groupoids, which contradicts *Pr.* 2.5. $\Diamond$

Thus the proof of *Th.* 2 is completed.

The following two propositions are corollaries of *Pr.* 1.6–1.7 and the definitions of $\mathcal{V}_{2,n}$ and $\mathcal{V}^{\square}_{2,n}$. We see from them that the condition $m \geq 3$ is essential for *Th.* 2.

PROPOSITION 2.12. *For every $n \geq 1$, $\mathcal{V}^{\square}_{1,n}$ is the variety of left–zero groupoids.* $\Diamond$.

PROPOSITION 2.13. *For every $n \geq 1$, $\mathcal{V}^{\square}_{2,n} = \mathcal{V}_{2,n}$.* $\Diamond$

3. $\mathcal{V}_{m,n}$–reduced sets

Assume that $\mathcal{V}$ is a (non–trivial) variety of groupoids, and $\mathbf{F} = (F, \cdot)$ a free groupoid with the basis B. Let $\approx_{\mathcal{V}}$ (furter on: $\approx$) be the least congruence on $\mathbf{F}$ such that $\mathbf{F}/_{\approx} \in \mathcal{V}$. If $u \in F$, then we denote by $u/_{\approx}$ the $\approx$–class containing u. We say that a subset S of F is $\mathcal{V}$*–reduced* iff the mapping $u \mapsto u/_{\approx}$ is a bijection from S onto $F/_{\approx}$. Thus:

PROPOSITION 3.1. *Let S be a $\mathcal{V}$–reduced set of F and the operation $\bullet$ is defined on S as follows:*

$$u,\, v,\, w \in S \Rightarrow (u \bullet v = w \Leftrightarrow uv \approx w)\,. \tag{3.1}$$

Then $u \mapsto u/_{\approx}$ is an isomorphism from $\mathbf{S} = (S, \bullet)$ onto $\mathbf{F}/_{\approx}$, and $\mathbf{S}$ is free in $\mathcal{V}$ with the basis B. $\Diamond$

PROPOSITION 3.2. *$R_{m,n}$ is $\mathcal{V}_{m,n}$-reduced set iff: $m=1$ or $m > n \geq 2$.*

Proof. If $m > n \geq 2$ or $m = 1$, then from *Th.* 1 and *Pr.* 1.6–*1.7* follows that $R_{m,n}$ is a $\mathcal{V}_{m,n}$–reduced set. Namely, the rewriting system (on $\mathbf{F}$) induced by elementary transformations: $u^m v^n \rightarrow \underline{uvm}$ is a terminating Church–Rose system ([5], 2.9), and $R_{m,n}$ consists of the normal forms in this system.

Let $m \geq 2$, $n = 1$ and $a \in B$.

If $m = 2$, then:

$$(a^2)^2 a \rightarrow a^2 a \rightarrow aa = a^2 \in R_{2,1}\,,$$
$$(a^2)^2 a = (a^2 a^2)a \rightarrow (aa^2)a \in R_{2,1}\,.$$

If $m \geq 3$, then:

$$(a^m)^m a \rightarrow \underline{a^m am} = a^m a\, \underline{a^m am-2} \rightarrow a^m a^m\, \underline{aa^m m-3} \rightarrow$$
$$\rightarrow \underline{aa^m m}\, \underline{aa^m m-3} \in R_{m,1}\,,$$

$$(a^m)^m a = a^m a^m \underline{a^m a^m m-2}\, a \to \underline{aa^m m}\,\underline{a^m a^m m-2}\, a \in R_{m,1}\,.$$

If $m = n \geq 2$, then:

$$(a^n)^{n+1} = (a^n)^n a^n \to \underline{a^n an} \in R_{n,n}\,,$$

$$(a^n)^{n+1} = a^n a^n \;\underline{a^n a^n n-1} \to a^n \;\underline{a^n a^n n-1} =$$
$$= (a^n)^n \to (a^n)^{n-1} \to \cdots \to a^n \in R_{n,n}\,.$$

Finally, if $2 \leq m < n$, then:

$$(a^n)^m (a^n)^n \to \underline{a^n a^n m} = (a^n)^m \in R_{m,n}\,,$$

$$(a^n)^m (a^n)^n = (a^n)^m \big((a^n)^m a^n\big)\, \underline{a^n a^n n-m-1} \to$$
$$\to (a^n)^m (\underline{a^n am}\,\underline{a^n a^n n-m-1}) \in R_{m,n}\,.$$

Therefore, if $m > n = 1$ or $2 \leq m \leq n$, then there exist $u, v \in R_{m,n}$ such that $u \neq v$, $u \approx v$, i.e. $R_{m,n}$ is not $\mathcal{V}_{m,n}$–reduced. $\diamondsuit$

From *Pr.* 3.2 follows that the definition of $R_{m,n}$ is "unsuccessful" if $m > n = 1$ or $2 \leq m \leq n$.

References

[1] R. H. Bruck: *A Survay of Binary Systems*, Berlin–Götingen–Heidelberg, 1956

[2] N. Celakoski: *On Semigroup Associatives*, Macedonian Academy of Sciences and Arts, Contributions IX 2 (1977), 5–19.

[3] R. M. Cohn: *Universal Algebra*, Harpers Series in Modern Mathematics, 1965.

[4] Ǵ Čupona, N. Celakoski: *Polyadic Subsemigroups of Semigroups*, Algebraic Conference Skopje 1980, 131–152.

[5] O. G. Kharalampovich, M. V. Sapir: *Algorithmic Problems in Varieties*, International Journal of Algebra and Computation, Vol. 5 (1995), 379–602.

[6] S. Markovski: *On quasivarieties of generalized subalgebras*, Algebraic Conference Skopje 1980, 125–129.

[7] M. D. Prešić, S. B. Prešić: *On the Embedding of Ω-algebras in Groupoids*, Publ. de l'Inst. Math. Nouv. sèrie, tome 21(35), 1977, 169–174.

[8] А. И. Мальцев: *Алгебраические системы*, Москва 1970.

СЛОБОДНИ ОБЈЕКТИ ВО НЕКОИ МНОГУОБРАЗИЈА ГРУПОИДИ

Ѓ. Чупона* и Н. Целакоски **

Р е з и м е

Во работава се дава каноничен опис на слободните објекти во многуобразието групоиди $x^m y^n = xyx\cdots$, каде што $m > n \geq 2$, а на десната страна се појавуваат m фактори, по ред: x, y, x, $y, \ldots$

* MANU

p. fah 428,

91 000 Skopje,

Makedonija

** Mašinski fakultet,

p. fah 464

91 000 Skopje,

Makedonija