Математички Билтен
20 (XX)
1996 (61–68)
Скопје, Македонија

ISSN 0351–336X

THE CENTER CONDITIONS FOR A CLASS OF CUBIC SYSTEMS

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Abstract

One of famous problems of the theory of Differential equations, namely, center–focus problem, are considered. For the cubic vector field

$$i\,\tfrac{dw}{dt} = w\left(1 - a_{10}w - a_{01}\overline{w} - a_{11}w\,\overline{w} - a_{-13}w^{-1}\,\overline{w}^3\right)\,,$$

in the case when $a_{11} \neq -a_{10}a_{01}$ the necessary and some sufficient center conditions are obtained.

We consider the cubic system of the form

$$i\,\frac{dw}{dt} = w\left(1 - a_{10}w - a_{01}\,\overline{w} - a_{11}w\,\overline{w} - a_{-13}w^{-1}\,\overline{w}^3\right)\,, \tag{1}$$

where $w = x + iy$, $a_{ij} \in \mathbb{C}$.

To solve the problem of distinguishing between a center and focus it is necessary to obtain Lyapunov focus quantities, which are polynomials of coefficients a_{ij}, $\overline{a}_{ij}$ of the system (1). There are many algorithms for its calculation (see, for example, [1], [3], [6], [7]), but we use the algorithm described in [4, 5]. Let us denote $x = (x_1, x_2, \ldots, x_8)$, $b_{ij} = \overline{a}_{ji}$,

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Mathematics subject classification: 34C

$$[x] = a_{10}^{x_1}\, a_{01}^{x_2}\, a_{11}^{x_3}\, a_{-13}^{x_4}\, b_{3-1}^{x_5}\, b_{11}^{x_6}\, b_{10}^{x_7}\, a_{01}^{x_8}\,.$$

Computing the function

$$V(x) = \sum_{x_1,\dots,x_8\geq 0} V_{(x_1,\dots,x_8)}\,[x]\,,$$

by the recurrent formula obtained in [4] we get Lyapunov focus quantities $g_{11}, g_{22}, g_{33}, \dots$. We denote by I the ideal generated by whole focus quantities, and by I_k the ideal generated bi k first quantities, i.e. $I = (g_{11}, g_{22}, \dots)$, $I_k = (g_{11}, \dots, g_{kk})$. After factoring every calculated quantity g_{kk} modulo the ideal I_{k-1} we obtain:

$$g_{11} = 2i(Im[0010\ 0000] + Im[1100\ 0000])\,,$$

$$g_{22} = Im[1100\ 0100]\ \mathrm{mod}(g_{11})\,,$$

$$g_{33} = -\frac{9}{4}\,Im[0300\ 1001] - \frac{5}{4}\,Im[0001\ 1011] + \frac{1}{4}\,Im[0100\ 1003] -$$
$$-\frac{9}{8}\,Im[0001\ 0040] - \frac{1}{8}\,Im[0200\ 1002]\ \mathrm{mod}\, I_2$$

where Im a is the imaginary part of complex number a.

Theorem 1. If the system (1), where $a_{11} \neq -a_{10}a_{01}$ has a center in the origin, then one of conditions holds:

(i) $a_{11} = Im\, a_{10}a_{01} = Im\, a_{01}^4\,\overline{a}_{-13} = Im\, a_{10}^4 a_{-13} = 0$;

(ii) $a_{10} = Im\, a_{11} = 0$;

(iii) $a_{01} = Im\, a_{11} = Im\, a_{10}^4\, a_{-13} = 0$;

(iv) $Im\, a_{10}a_{01}\overline{a}_{11} = Im\, a_{10}a_{01} = Im\, a_{01}^4\,\overline{a}_{-13} = 0$
$a_{10} \neq 0,\ a_{01} \neq 0,\ a_{11} \neq 0$;

(v) $Im\, a_{11} = a_{10} - \frac{1}{2}\,\overline{a}_{01} = 0$;

(vi) $a_{10} - 3a_{01} = |a_{-13}| - 2\,|a_{01}|^2 = 0$.

Proof. From the second center condition $g_{22} = 0$ obtain

$$a_{10}a_{01}b_{11} = h\,, \qquad (2)$$

where $h \in \mathbf{R}$. Therefore we can consider three cases:

(a) $a_{10} = 0$;

(b) $a_{01} = 0$;

(c) $a_{10} \neq 0,\ a_{01} \neq 0$.

(a) In this case from the condition $g_{11=0}$ we have $Im\, a_{11} = 0$ and, hence, the condition (ii) is fulfilled.

(b) The condition $g_{11} = 0$ yields $Im\, a_{11} = 0$. Computing the next polynomials g_{ii} under the condition $a_{01} = 0$ we obtain

$$g_{22} \equiv \text{ mod } I_1\,, \quad g_{33} \equiv 0 \text{ mod } I_1\,, \quad g_{44} \equiv Im[0010\ 1004] \text{ mod } I_1\,.$$

Hence, from the equation $q_{44} = 0$ we get either

$$a_{11} = 0$$

or

$$Im\, a_{10}^4 a_{-13} = 0\,.$$

Thus we have either the condition (iii) or

$$a_{01} = a_{11} = 0\,.$$

In the last case, computing focus quantity we get

$$g_{55} = Im[0001\ 2004]\,,$$

i.e. the equation $g_{55} = 0$ yields

$$|a_{-13}|^2\ Im\, a_{10}^4 a_{-13} = 0\,,$$

and therefore the conditions (i), (iii) holds.

(c) In this case from (2) we obtain

$$a_{11} = g\, a_{10} a_{01}\,, \tag{3}$$

where $g \in \mathbf{R}$, and then from the equation $g_{11} = 0$

$$Im[1100\,,0000]\,(1 + g) = 0\,. \tag{4}$$

In this paper we consider only the case $g \neq -1$ (i.e. $a_{11} \neq -a_{10}a_{01}$) and therefore, taking into account that $a_{01} \neq 0$, from eq. (4) we obtain

$$a_{10} = s\, \overline{a}_{01} \tag{5}$$

where $s \in \mathbf{R}$.

Substituting this expression to the equation $g_{33} = 0$ we get

$$Im[0400\ 1000]\left(\frac{1}{4}s^3 - \frac{1}{8}s^2 - \frac{9}{4}s + \frac{9}{8}\right) = 0\,.$$

Therefore we have either the condition (iv) or one of the next three conditions:

$$(\alpha)\quad s = 3\,,$$

$$(\beta)\quad s = -3\,,$$

$$(\gamma)\quad s = \frac{1}{2}\,.$$

It was mentioned above that the calculation of Lyapunov focus quantities is very difficult computational problem. To simplify calculations we recalculate polynomials g_{ii} for every from cases $(\alpha) - (\gamma)$ analogously as we had done in [5].

Computing we obtain in the corresponding cases.

(α) In this case from $g_{11} = 0$ we have $a_{11} = \alpha$, where $\alpha \in \mathbb{R}$, and

$$g_{22} = 0 \bmod I_1\,,\quad g_{33} = 0 \bmod I_1\,,$$

$$g_{44} = -\frac{70}{3}Im[0101\ 0050] - \frac{35}{3}Im[0011\ 0040] \bmod I_1\,.$$

Hence, using the correlation $g_{44} = 0$ we have

$$Im[0001\ 0040]\left(-\frac{70}{3}|a_{01}|^2 - \frac{35}{3}\alpha\right) = 0\,.$$

Thus we get either the condition (iv) or

$$a_{11} = -2a_{01}b_{10} = -2|a_{01}|^2\,.$$

In the last case computing fifth Lyapunov focus quantity we obtain

$$g_{55} = -\frac{25}{9}Im[0002\ 1040] + \frac{100}{9}Im[0201\ 0060]\,.$$

Equation this polynomial to zero we have

$$Im[0001\ 0040]\left(-\frac{25}{9}|a_{-13}|^2 + \frac{100}{9}|a_{01}|^4\right) = 0$$

Thus either the condition (iv) or (vi) takes place.

(β) In this case computing sixth focus quantity we have

$$g_{66} \equiv Im[0031\ 0040] \bmod I_5 \,.$$

Therefore the conditions $g_{66} = 0$ yields

$$\alpha^3\, Im[0001\ 0040] = 0 \,.$$

and we get either (iv) or $a_{11} = 0$.

The case $a_{11} = 0$ we will consider below.

(γ) In this case, obviously, we have the condition (v).

There remains to consider the case when $a_{11} = 0$. Then the equation $g_{11} = 0$ implies

$$Im\, a_{10} a_{01} = 0 \,,$$

and, therefore, it is necessary to consider two cases:

$$(\alpha\alpha) \quad a_{01} = 0 \,,$$

$$(\beta\beta) \quad a_{01} \neq 0 \,.$$

($\alpha\alpha$) This case we have considered above.

($\beta\beta$) In case the correlation (5) is fulfilled, and, computing g_{44} and equaling it to zero we get

$$s^3 \left(-\frac{28}{27}\, s + \frac{14}{27} \right) Im[0001\ 0040]\ [0100\ 0010] = 0 \,.$$

Therefore one of conditions is realized:

$$s = \frac{1}{2} \qquad \text{(i.e. (v))} \,.$$

or

$$Im[0001\ 0040] = 0 \,.$$

In the last case we obtain

$$Im[4001\ 0000] = s^4\, Im[0001\ 0040] = 0 \,. \tag{6}$$

Thus the condition (i) is fulfilled.

Remark 1. The fact that (i)–(iii) are the necessary center conditions has been established in [2], by we have obtained it by the independent method.

Theorem 2. The conditions (i)–(iv) are the sufficient center conditions for the system (1).

Proof. The sufficiency of the conditions (i)–(iii) are proved in [2].

In the case (iv) using the correlations (3), (5) and $a_{-13} = ta_{01}^4 \quad (t \in \mathbf{R})$, which is the corollary of the equation

$$Im[0001\,0040] = 0\,,$$

we can conclude that polynomials g_{ii} are polynomials only on partitions of vectors (0, 1), (1, 0), and therefore from the structure of polynomials g_{kk} [4, 5] we obtain that $g_{kk} \equiv 0\ \forall\, i$, i.e. the corresponding system (1) has a center.

Remark 2. It seems the conditions (v), (vi) also are the sufficient center conditions, because according to our calculations, in the case (vi) $g_{ii} \equiv 0 \quad \forall\, i = \overline{1,12}$ and in the case (v) $g_{ii} = 0 \quad \forall\, i = \overline{1,10}$. But to prove this fact it is necesary to find a holomorphic integral.

Remark 3. In the case $a_{11} = -a_{10}a_{01}$ we have computed seven polynomials $g_{11} \ldots, g_{77}$, but the polynomials $g_{55}, \ldots, g_{77}$ contain about $20 - 30$ terms and the problem is to obtain a more simple description of the zero set of the ideal $I = (g_{11}, \ldots, g_{77})$.

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УСЛОВИ ЗА ЦЕНТАР ЗА ЕДНА КЛАСА СИСТЕМИ

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Резиме

Се посматра еден од познатите проблеми на теоријата на диференцијалните равенки, имено проблемот на центар–фокус. За кубното векторско поле

$$i\,\frac{dw}{dt} = w\left(1 - a_{10}w - a_{01}\,\overline{w} - a_{11}w\,\overline{w} - a_{-13}w^{-1}\,\overline{w}^{3}\right)$$

во случајот кога $a_{11} \neq -a_{10}a_{01}$ се добиени потребните, и некои доволни услови за центар.

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