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# SHARPENINGS OF JENSEN'S INEQUALITY AND APPLICATIONS

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**Abstract**

Some new refinements of the well known Jansen's inequality and certain natural applications are given

## 1. Introduction

The following inequality is well known in literature as Jansen's inequality

$$f\Big(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\Big) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \tag{1}$$

where $f$: $C \subseteq X \to R$ is a convex mapping on the convex subset $C$ of the real linear space $X$, $x_I$ are in $C$ $(i = 1, 2, \ldots, n)$ and $p_i \ge 0$ with

$$P_n := \sum_{i=1}^{n} p_i > 0\,.$$

In the recent paper [7] is considered the following sequences of mappings:

$$F_1^{[m]}(t) := \frac{1}{P_n} \sum_{i_1=1}^{n} p_{i_1} f\Big[\alpha_1(t)x_{i_1} + \alpha_2(t)x_{i_2} + \cdots + + (\alpha_m(t))\frac{1}{P_n} \sum_{i_1=1}^{n} p_i x_i\Big]$$

$$F_2^{[m]}(t) := \frac{1}{P_n^2} \sum_{i_1,i_2=1}^{n} p_{i_1} p_{i_2} f\Big[\alpha_1(t)x_{i_1} + \alpha_2(t)x_{i_2} + \cdots + + \Big(\alpha_3(t) + \cdots + \alpha_m(t)\Big)\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\Big]$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$F_{m-1}^{[m]}(t) := \frac{1}{P_n^{m-1}} \sum_{i_1,\ldots,i_{m-1}=1}^{n} p_{i_1} \cdots p_{i_{m-1}} f\Big[\alpha_1(t)x_{i_1} + \cdots + + \alpha_{m-1}(t)x_{i_{m-1}} + \alpha_m(t)\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\Big]$$

and

$$F^{[m]}(t) := \frac{1}{P_n^m} \sum_{i_1,\ldots,1_m=1}^{n} p_{i_1} \cdots p_{i_m} f\Big(\alpha_1(t)x_{i_1} + \cdots + \alpha_m(t)x_{i_m}\Big)$$

where $\alpha_1, \ldots, \alpha_m \colon T \to R$ are $m$ functions with the property that $\alpha_i(t) \geq 0$ and $\alpha_1(t) + \cdots + \alpha_m(t) = 1$ for all $t \in T\Big(i = \overline{1, m}\Big)$, and is proved, between other, the following refinement of Jensen's result:

$$f\Big(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\Big) \leq F_1^{[m]}(t) \leq \cdots \leq F_{m-1}^{[m]}(t) \leq F^{[m]}(t) \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i).$$

For other sharpenings of this classic fact see also the papers [7] and [9].

## 2. The Main Result

Now, we will point out the main result of this paper which give another refinement of (1).

**Theorem.** *Let $f$, $x_i$, $p_i$, $k$ be as above and $e_s^{(j)}$ are real numbers with $e_s^{(1)} + \cdots + e_s^{(k)} = 1$ for all $s = 1, \ldots, m$ ($m$ is a given natural number). Suppose $w_s \geq 0$ ($s = 1, \ldots, m$) with $W_m := \sum_{s=1}^{m} w_s > 0$. Then one has the inequalities:*

$$
\begin{aligned}
f\Big(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\Big) &\leq \frac{1}{P_n^k} \sum_{i=1,\ldots,i_k=1}^{n} p_{i_1}\cdots p_{i_k} \times \\
&\quad \times\ f\Big[\Big(\frac{1}{W_m}\sum_{s=1}^{m} e_s^{(1)} w_s\Big) x_{i_1} + \cdots + \Big(\frac{1}{W_m}\sum_{s=1}^{m} e_s^{(k)} w_s\Big) x_{i_k}\Big] \leq \\
&\leq \frac{1}{P_n^k W_m} \sum_{i=1,\ldots,i_k=1}^{n} p_{i_1}\cdots p_{1_k} \times \\
&\quad \times \Big[\sum_{s=1}^{m} w_s f\Big(e_s^{(1)} x_{i_1} + \cdots + e_s^{(k)} x_{i_k}\Big)\Big] \leq \\
&\leq \frac{1}{P_n} \sum p_i f(x_i)\,.
\end{aligned}
\tag{2}
$$

**Proof.** By Jensen's inequality for multiple sums, we have:

$$
\begin{aligned}
&\frac{1}{P_n^k} \sum_{i_1,\ldots,i_k=1}^{n} p_i \cdots p_{i_k} \times \\
&\times f\Big[\Big(\frac{1}{W_m}\sum_{s=1}^{m} e_e^{(1)} w_s\Big) x_{i_1} + \cdots + \Big(\frac{1}{W_m}\sum_{s=1}^{m} e_s^{(k)} w_s\Big) x_{i_k}\Big] \geq \\
&\geq f\Big(\frac{1}{P_n^k} \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\cdots p_{i_k} \Big[\Big(\frac{1}{W_m}\sum_{s=1}^{m} e_s^{(1)} w_s\Big) x_{i_1} + \cdots + \\
&\quad + \Big(\frac{1}{W_m}\sum_{s=1}^{m} e_s^{(k)} w_s\Big) x_{i_k}\Big]\Big)\,.
\end{aligned}
$$

Since a simple computation shows that:

$$\frac{1}{P_n^k}\sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\cdots p_{i_k}\times$$

$$\times\, f\Big[\Big(\frac{1}{W_m}\sum_{s=1}^{m} e_s^{(1)}w_s\Big)x_{i_1}+\cdots+\Big(\frac{1}{W_m}\sum_{s=1}^{n} e_s^{(k)}w_s\Big)x_{i_k}\Big]=$$

$$=\frac{1}{W_m}\sum_{s=1}^{m} e^{(1)}w_s\,\frac{1}{P_n^k}\sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\dots p_{i_k}x_{i_1}+\cdots+$$

$$+\frac{1}{W_m}\sum_{s=1}^{m} e_s^{(k)}w_s\,\frac{1}{P_n^k}\sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\cdots p_{i_k}x_{i_k}=$$

$$=\frac{1}{W_m}\sum_{s=1}^{m} e_s^{(1)}w_s\,\frac{1}{P_n}\sum_{i=1}^{n} p_ix_i+\cdots+\frac{1}{W_m}\sum_{s=1}^{m} e_s^{(k)}\,\frac{1}{P_n}\sum_{i=1}^{n} p_ix_i=$$

$$=\frac{1}{P_n}\sum_{i=1}^{n} p_ix_i\,\frac{1}{W_m}\sum_{s=1}^{m}\Big(e_s^{(1)}+\cdots+e_s^{(k)}\Big)w_s=$$

$$=\frac{1}{P_n}\sum_{i=1}^{n} p_ix_i\,.$$

the first of (2) is thus proven.

By Jensen's inequality, we also have:

$$\frac{1}{W_m}\sum_{s=1}^{m} w_sf\Big(e_s^{(1)}x_{i_1}+\cdots+e_s^{(k)}x_{i_k}\Big)\geq$$

$$\geq f\Big(\frac{1}{W_m}\sum_{s=1}^{m} w_s\Big(e_s^{(1)}x_{i_1}+\cdots+e_s^{(k)}x_{i_k}\Big)\Big)=$$

$$=f\Big[\Big(\frac{1}{W_m}\sum_{s=1}^{m} e_s^{(1)}w_s\Big)x_{i_1}+\cdots+\Big(\frac{1}{W_m}\sum_{s=1}^{m} e_s^{(k)}w_s\Big)x_{i_k}\Big]\,.$$

By multiplying this inequality with $p_{i_1}\cdots p_{i_k}\geq 0$ and summing after $i_1,\ldots,i_k$ to 1 at $n$, we derive the second inequality in (2).

Now, let observe that:

$$f\left(e_s^{(1)}x_{i_1}+\cdots+e_s^{(k)}x_{i_k}\right)\leq e_s^{(1)}f(x_{i_1})+\cdots+e_s^{(k)}f(x_{i_k})$$

for all $s=1,\ldots,m$. If we multiply this inequality with $w_s\geq 0$ and summing after $s$ to 1 at $m$, we conclude:

$$\frac{1}{W_m}\sum_{s=1}^{n} w_s f\left(e_s^{(1)}x_{i_1}+\cdots+e_s^{(k)}x_{i_k}\right)\leq$$

$$\leq\frac{1}{W_m}\sum_{s=1}^{m} w_s e_s^{(1)}f(x_{i_1})+\cdots+\frac{1}{W_m}\sum_{s=1}^{m} w_s e_s^{(k)}f(x_{i_k}).$$

Finally, by multiplying this inequality with $p_{i_1}\ldots p_{i_k}\geq 0$ and summing after $i_1,\ldots,i_k$ to 1 at $n$, we deduce:

$$\frac{1}{P_n^k W_m}\sum_{i_1,\ldots,i_k=1}^{n}\left[\sum_{s=1}^{m} w_s f\left(e_s^{(1)}x_{i_1}+\cdots+e_s^{(k)}x_{i_k})\right)\right]\leq$$

$$\leq\frac{1}{W_m}\sum_{s=1}^{m} w_s\,e_s^{(1)}\,\frac{1}{P_n^k}\sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\ldots p_{i_k}f(x_{i_1})+\cdots+$$

$$+\frac{1}{W_m}\sum_{s=1}^{m} w_s\,e_s^{(k)}\,\frac{1}{P_n^k}\sum_{i_1,\ldots,i_k=1}^{n}\ldots p_{i_k}f(x_{i_k})=$$

$$\frac{1}{W_m}\sum_{s=1}^{m} w_s^{(1)}\,\frac{1}{P_n}\sum_{s=1}^{n} p_i f(x_i)+\cdots+\frac{1}{W_m}\sum_{s=1}^{m} w_s\,e_s^{(k)}\,\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i)=$$

$$=\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i),$$

and the proof is finished.

The following corollary also holds.

**Corollary**. Let $f\colon C \subseteq X \to (0, \infty)$ be a convex mapping on convex set $C$ of real linear space $X$ which is also logaritmically concave on $C$, i.e., the mapping $\log f$ is concave on $C$, then for all $x_i$, $p_i$, $e_s^{(j)}$, $w_s$ and $m$ as above, we have the following improvement of arithmetic mean – geometric mean inequality:

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \geq \frac{1}{P_n^k W_m} \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\ldots p_{i_k} \times$$

$$\times \Big[\sum_{i=1}^{m} w_s f\Big(e_s^{(1)}x_{i_1} + \cdots + e_s^{(k)}x_{i_k}\Big)\Big] \geq$$

$$\geq \frac{1}{P_n^k} \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\ldots p_{i_k} \times$$

$$\times f\Big[\Big(\frac{1}{W_m}\sum_{i=1}^{m} e_s^{(1)} w_s\Big)x_{i_1} + \cdots + \Big(\frac{1}{W_m}\sum_{s=1}^{m} e_s^{(k)} w_s\Big)x_{i_k}\Big] \geq$$

$$\geq f\Big(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\Big) \geq \tag{3}$$

$$\geq \Big\{\prod_{i_1,\ldots,i_k=1}^{n} f\Big[\Big(\frac{1}{W_m}\sum_{s=1}^{m} e_s^{(1)} w_s\Big)x_{i_1} + \cdots +$$

$$+ \Big(\frac{1}{W_m}\sum_{s=1}^{m} e_s^{(k)} w_s\Big)x_{i_k}\Big]^{p_{i_1}\ldots p_{i_k}}\Big\}^{1/P_n^k} \geq$$

$$\geq \Big\{\prod_{i_1,\ldots,i_k=1}^{n} \Big[\prod_{s=1}^{n} f^{w_s}\Big(e_s^{(1)}x_{i_1} + \cdots + e_s^{(k)}x_{i_k}\Big)\Big]^{p_{i_1}\ldots p_{i_k}}\Big\}^{1/(P_n^k W_m)} \geq$$

$$\geq \Big[\prod_{i=1}^{n} f^{p_i}(x_i)\Big]^{1/P_n}.$$

**Proof.** The argument of the second part of (3) follows from (2) for the convex mapping $-\log f$, and we omit the details.

**Remark.** If in the above theorem we chosse $m = 1$, $e_1^{(1)} = q_1 | Q, \ldots, e_1^{(k)} = q_k | Q_k$, where $q_j \geq 0$, $j = 1, \ldots, k$ and $Q_k > 0$ $(1 \leq k \leq n)$, then we recapture the improvement of Jensen's inequality embodied in Theorem.

## 3. Applications

**1.** Let $\left(X, \|\cdot\|\right)$ be a real normed space, $x_i \in X$ and $p_i$, $k$, $e^{(j)}$, $w_s$ and $m$ be as in the above theorem. Then for all $p \geq 1$ we have the inequalities:

$$P_n^{p-1} \sum_{i=1}^{n} p_i \| x_i \|^p \geq \frac{P_n^{p-k}}{W_m} \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1} \cdots p_{i_k} \times$$

$$\times \left[ \sum_{s=1}^{m} w_s \| e_s^{(1)} x_{i_1} + \cdots + e_s^{(k)} x_{i_k} \|^p \right] \geq$$

$$\geq P_n^{p-k} \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1} \cdots p_{i_k} \| \left( \frac{1}{W_m} \sum_{s=1}^{m} e_s^{(1)} w_s \right) x_{i_1} + \cdots +$$

$$+ \left( \frac{1}{W_m} \sum_{s=1}^{m} e_s^{(k)} w_s \right) x_{i_k} \|^p \geq \| \sum_{i=1}^{n} p_i x_i \|^p .$$

The proof follows from inequality (2) for the convex mapping $f: X \to R$, $f(x) = \| x \|^p$.

**2.** Now. let consider the mapping $f: (0, 1/2] \to (0, \infty)$ given by $f(x) = \left(\frac{x}{1-x}\right)^r$, $r \geq 1$. It is easy to see that $f$ is convex on $(0, 1/2)$ and also logarithmically concave on this interwal. By the use of the above corollary one gets:

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i \left( \frac{x_i}{1 - x_i} \right)^r \geq \frac{1}{P_n^k W_m} \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1} \cdots p_{i_k} \left[ \sum_{s=1}^{n} w_s \times \right.$$

$$\left. \times \left[ \frac{e_s^{(1)} x_{i_1} + \cdot + e_s^{(1)} x_{i_k}}{e_s^{(1)}(1 - x_{i_1}) + \cdots + e_s^{(k)}(1 - x_{i_k})} \right)^r \right] \geq$$

$$\geq \frac{1}{P_n^k} \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1} \cdots p_{i_k} \times$$

$$\times\left[\frac{\left(\sum_{s=1}^{m} e_s^{(1)} w_s\right) x_{i_1} + \cdots + \left(\sum_{s=1}^{m} e_s^{(k)} w_s\right) x_{i_k}}{\left(\sum_{s=1}^{m} e_s^{(1)} w_s\right)(1 - x_{i_1}) + \cdots + \left(\sum_{s=1}^{m} e_s^{(k)} w_s\right)(1 - x_{i_k})}\right]^r \geq$$

$$\geq \left[\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i (1 - x_1)}\right]^r \geq$$

$$\geq \left\{\prod_{i_1,\ldots,i_k=1}^{n}\left[\left(\frac{\left(\sum_{s=1}^{m} e_s^{(1)} w_s\right) x_{i_1} + \cdots + \left(\sum_{s=1}^{m} e_s^{(k)} w_s\right) x_{i_k}}{\left(\sum_{s=1}^{m} e_s^{(1)} w_s\right)(1 - x_{i_1}) + \cdots + \left(\sum_{s=1}^{m} e_s^{(k)}\right)(1 - x_{i_k})}\right)^r\right]^{p_{i_1}\cdots p_{i_k}}\right\}^{\frac{1}{P_n^k}} \geq$$

$$\geq \left\{\prod_{i_1,\ldots,i_k=1}^{n}\left[\prod_{s=1}^{m}\left(\frac{e_s^{(1)} x_{i_1} + \cdots + e_s^{(k)} x_{i_k}}{e_s^{(1)}(1 - x_{i_1}) + \cdots + e_s^{(k)}(1 - x_{i_k})}\right)^{r w_s}\right]^{p_{i_1}\cdots p_{i_k}}\right\}^{1/(P_k^k W_m)} \geq$$

$$\geq \left[\frac{\prod_{i=1}^{n} x_i^{p_i}}{\prod_{i=1}^{n} (1 - x_i) p_i}\right]^{1/P^n}$$

where $x_i \in (0, 1/2]$ $(i = 1, \ldots, n)$ and $p_i$ $e_s^{(j)}$, $w_s$, $k$ and $m$ are as above.

**Remark.** The previous inequalites contain refinements of C–L. Wang's inequality [12] and also (if $p_i = 1,\ i = 1, \ldots, n$) of Ky Fan' s result [3, p. 5].

## References


[1] A. Andrica and I. Rasa: *The Jensen's inequlity: refinements and applications*, L'Anal. Num. Theor. L'Approx, 14 (1985), 105–108.

[2] R. E. Barlow, A. W. Marshall and F. Praschan: *Some inequalities for starshaped and convex functions*, Pacific J. Math., 29 (1969), 19–42.

[3] E. F. Breckembach and R. Bellman: *Inequalities*, 4th ed. Springer, Berlin, 1983.

[4] S. S. Dragomir: *A refinement of Jensen inequality*, G. M. Metod. (Bucharest), 10 (1989), 190–191.

[5] S. S. Dragomir and N. M. Ionescu: *On some inequalities for convex-dominated functions*, L'Anal. Num. Theor. L' Approx., 19 (1990), 21–28.

[6] S. S. Dragomir: Some refinements of Ky Fans inequality, J. Math. Anal. Appl. Vol. **163**, No.2 (1992), 317–321.

[7] S. S. Dragomir and D. M. Milošević: *A sequences of mappings conected with Jensen's inequality and applications*, Mat. Vesnik, **44** (1992), 113–121.

[8] J. E. Pečarić: *Inverse of Jensen–Steffanse's inequality*, Glasnik Mat., **16** (1981), 229–233.

[9] J. E. Pečarić and S.S. Dragomir: *A refinement of Jensen inequality and Applications*, Studia Unio. Babes–Bolyai, Mathematica (1), **34** (1989), 15–19.

[10] T. Popoviciu: *Sur une inégalité de N. Levinson*, Mathematica, **6** (1964), 301–306.

[11] P.M. Vasić and Z. Mijalković, on an index set function connected with Jensen inequality, Unio, Beograd, Publ. Elektr. Fak. Ser. Mat:Fiz., No. 544 - No.576, (1976), 110–112.

[12] C. W. Wang: *An a Ky Fan Inequality of the complementary A–G type and its variants*, J. Math, Anal. Appl., **73** (1980), 501–505.

# ПОДОБРУВАЊЕ НА НЕРАВЕНСТВОТО НА ЈЕНСЕН И ПРИМЕНИ

Дадени се некои нови подобрувања на добро познатото неравенство на Јенсен и некои природни примени кои од тоа произлегуваат.

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