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ABOUT FINAL GROUPS OF TRANSFORMATIONS OF THE NONLINEAR DIFFERENTIAL EQUATION OF THE SECOND ORDER

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Abstract

In the given work, the final groups of transformations of the equations of a type

$$y'' = \alpha x^n y^m y'^l \tag{1}$$

are considered allowing to divide a class of the equations (1) on not crossing final subclasses, inside which the solution of one representative of the given subclass guarantees to us the solutions of all other representatives of the same subclass with the help of known transformations.

The equation (1) is generalized homogeneous, therefore its order can easily be decreased by one by transformation:

$$x = e^t\,, \quad y = u\,e^{\frac{n-l+2}{m+l-1}t}\,; \quad u' = p \quad (u'' = pp')\,.$$

After replacement

$$\xi = \frac{2n+m-l+3}{m+l-1}\,u$$

we receive the equation

$$\begin{aligned} pp' - p = &- \frac{(n-l+2)(m+n+1)}{(2n+m-l+3)^2}\,\xi \\ &+ A\xi^m \left(p - \frac{n-l+2}{2n+m-l+3}\,\xi\right)^l . \end{aligned} \tag{2}$$

In a case $2n+m-l+3 \neq 0$, $m+l-1 \neq 0$ is generalization of the equation Abel of 2nd order

$$pp' - p = R(\xi)\,.$$

Let's consider formal transformations of the initial equation (1), taking in to account, that in each concrete case the solution should be checked up. Symbolically, the initial equation will be written as (m, n, l).

1. Let y be independent variable, than we have

$$x'' = -ay^m x^n x'^{3-l}$$

thus we receive transition $(m, n, l) \to (n, m, 3-l)$. Let's designate this transformation by the letter r.

2. We shall transform the equation of a type (1) with parameters (μ, ν, λ) as follows. We multiply both parts of the equation on $y^{-\mu} y'^{-\lambda}$, raise to the power both parts of the equation in a degree $\dfrac{1}{\nu}$ and than differentiate:

$$yy'y^m - \lambda yy''^2 - \mu y'^2 y'' = \nu a^{\frac{1}{\nu}}\, y^{\frac{\mu+\nu}{\nu}}\, y'^{\frac{\lambda+\nu}{\nu}}\, y''^{\frac{\nu-1}{\nu}}\,.$$

After transformation

$$y' = z\,, \quad y = e^t\,, \quad z = w\, e^{\frac{\mu+\nu+1}{\nu-\lambda+2}\, t}\,, \quad w' = q\,, q = pw^{\lambda-1}$$

replacing

$$\xi = \frac{\mu\lambda + 2\nu\lambda + \mu\nu - 2\mu + \lambda - 3\nu - 2}{(\lambda-2)(\lambda-\nu-2)}\, w^{2-\lambda}$$

we receive

$$pp' - p = -\frac{\nu(\mu+\lambda-1)(\mu+\nu+1)(\lambda-2)}{(\mu\lambda+2\nu\lambda+\mu\nu-2\mu-3\nu-2)^2}\,\xi \tag{3}$$

$$+ A\xi^{\frac{1}{\lambda-2}} \left[p - \frac{(\lambda-2)(\mu+\nu+1)}{\mu\lambda+2\nu\lambda+\mu\nu-2\mu+\lambda-3\nu-2}\,\xi\right]^{\frac{\nu-1}{\nu}}.$$

We again have received the equation of a type (2). Let's require, that the factors and parameters of the equations (2) and (3) coincided, then we shall receive transition $(m, n, l) \to (\mu, \nu, \lambda)$. The solution of system of the algebraic equations gives

$$\mu = -\frac{n}{n+1}, \quad \nu = \frac{1}{1-l}, \quad \lambda = \frac{2m+1}{m}$$

i.e.

$$(m, nl) \to \left(-\frac{n}{n+1}, \frac{1}{1-l}, \frac{2m+1}{m}\right).$$

Designating this transformation by the letter f, and applying it repeatedly, we shall receive $\left(\frac{1}{l-2}, -\frac{m}{m+1}, \frac{n-1}{n}\right)$, we receive groups C_3.

3. Let in the initial equation $l = 0$:

$$y'' = ax^n y^m \qquad (m, n, 0). \tag{4}$$

In this special case (equation Emden-Fowler) transformation

$$x = x_1^{-1}, \quad y = \frac{y_1}{x_1}$$

gives

$$y_1'' = ax_1^{-m-n-3} y_1^m$$

i.e. $(m, n, 0) \leftrightarrow (m, -m-n-3, 0)$. Let's designate this transformation by the letter s. As $fs = sf$, the group represents $C_3 \times C_2$ i.e. isomorphic to C_5. We receive groups of transformations of the equation Emden-Fowler:

$$(m, n, 0) \to (m, -m-n-3, 0)$$
$$\to \left(-\frac{n}{n+1}, 1, \frac{2m+1}{m}\right) \to \left(-\frac{1}{2}, -\frac{m}{m+1}, \frac{m+n+4}{m+n+3}\right)$$
$$\to \left(-\frac{m+n+3}{m+n+2}, 1, \frac{2m+1}{m}\right) \to \left(-\frac{1}{2}, \frac{m}{m+1}, \frac{n-1}{n}\right).$$

4. We shall make in the initial equation (1) following transformations multiplying both parts of the equation on $x^{-n}y^{-m}y'^{l}$ and differentiate. Further, we put

$$x = e^t, \quad y = e^{\int u\, dt}, \quad u' = q, \quad q = pu^l$$

and receive

$$pp' - (m+2l-3)u^{1-l}p - (n-2l+3)u^{-1}p =$$

$$= -(1-l-m)u^{3-2l} - (m+2l-n-3)u^{2-2l} - (2-l+n)u^{1-2l}\,.$$

Accordingly, from (μ,ν,λ) we shall receive

$$pp' - (\mu+2\lambda-3)u^{1-\lambda}p - (\nu-2\lambda+3)u^{-\lambda}p =$$

$$= -(1-\lambda-\mu)u^{3-2\lambda} - (\mu+2\lambda-\nu-3)u^{2-2\lambda} - (2-\lambda+\nu)u^{1-2\lambda}\,.$$

In the first equation let's put

$$\xi = \frac{m+2l-3}{2-l}\,u^{2-l},$$

in second

$$\xi = \frac{\nu-2\lambda+3}{1-\lambda}\,u^{1-\lambda},$$

therefore the equation transform in

$$pp' - p = \xi^{\frac{1}{l-2}}\left[p - \frac{(n-2l-m+3)(l-2)}{(n-2l+3)(m+2l-3)}\,\xi\right] - \frac{(l+m-1)(l-2)}{(m+2l-3)^2}\,\xi - \frac{(l-n-2)(l-2)}{(n-2l+3)^2}\,\xi^{\frac{l}{l-2}} \tag{5}$$

$$pp' - p = \xi^{\frac{1}{1-\lambda}}\left[p - \frac{(\nu-2\lambda-\mu+3)(\lambda-1)}{(\nu-2\lambda+3)(\mu+2\lambda-3)}\,\xi\right] - \frac{(\lambda+\mu-1)(\lambda-1)}{(\mu+2\lambda-3)^2}\,\xi^{\frac{3-\lambda}{1-\lambda}} - \frac{(\lambda-\nu-2)(\lambda-1)}{(\nu-2\lambda+3)^2}\,\xi\,. \tag{6}$$

Equating of the appropriate factors and parameters of the equations (5) and (6) gives not trivial solution only at

$$l = \frac{m+2n+3}{m+n+2}$$

$$\left(m,\, n,\, \frac{m+2n+3}{m+n+2}\right) \leftrightarrow \left(-\frac{m}{m+n+1},\, -\frac{n}{m+n+1},\, \frac{2m+n+3}{m+n+2}\right).$$

The description of transformations can be applied for research of the equation Emden-Fowler and some other nonlinear equations of physics, which is type (1) or reduced to it, and also allowing to find the large number of the equations of a type (1) and the equations Abel of 2nd order, integrated in quadratures and through known special functions.

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