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ABOUT TWO CLASSES OF ANALYTIC FUNCTIONS

Nikola Tuneski

Abstract

Let $\mathcal{A}$ be the class of analytic functions in the unit disk $\mathcal{U} = \{z : |z| < 1\}$ normalized such that $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Classes

$$M'_{\alpha,\rho} = \left\{ f \in \mathcal{A} : \left| f'(z) - \alpha f(z)/z + \alpha - 1 \right| < \rho, z \in \mathcal{U} \right\}$$

and

$$M''_{\alpha,\rho} = \left\{ f \in \mathcal{A} : \left| zf''(z) - \alpha f'(z) + \alpha \right| < \rho, z \in \mathcal{U} \right\}$$

were studied earlier by Fournier–Mocanu and Ponnusamy–Singh. In this paper, sharp upper bound of the Fekete–Szegö functional over these two classes is obtained. Also, sufficient conditions that embed this classes in the class of starlike functions of order α and in the class of convex functions order α are given.

1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of analytic functions in the unit disk $\mathcal{U} = \{z : |z| < 1\}$ normalized such that $f(0) = f'(0) - 1 = 0$, i.e., of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Further, for $0 \le \alpha < 1$ we say that $f \in \mathcal{A}$ is in the class of *starlike functions of order* α, denoted by $S^*(\alpha)$, if

$$\text{Re}\left[\frac{zf'(z)}{f(z)}\right] > \alpha,$$

$z \in \mathcal{U}$, and in the class of *convex functions of order* α, $K(\alpha)$, if

$$\text{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > \alpha,$$

$z \in \mathcal{U}$. Specially, $S^* \equiv S^*(0)$ is the class of *starlike functions* and $K \equiv K(0)$ is the class of *convex functions*.

Further, for $\alpha \in \mathbf{C}$ and $\rho > 0$ we define classes

$$M'_{\alpha,\rho} = \left\{ f \in \mathcal{A} : \left| f'(z) - \alpha \cdot \frac{f(z)}{z} + \alpha - 1 \right| < \rho, z \in \mathcal{U} \right\}$$

and

$$M''_{\alpha,\rho} = \{ f \in \mathcal{A} : |zf''(z) - \alpha f'(z) + \alpha| < \rho, z \in \mathcal{U} \} .$$

In [2], Fournier and Mocanu found the values of $\rho_1(\alpha) \equiv \sup \{\rho : M'_{\rho,\alpha} \subset S^*\}$ and $\rho_2(\alpha) \equiv \sup \{\rho : M''_{\rho,\alpha} \subset S^*\}$. Independently, Ponnusamy and Singh in [6] gave criteria for embedding $M''_{\rho,\alpha}$ in the class of univalent functions and some of its subclasses. In this paper we continue investigation of the classes $M'_{\rho,\alpha}$ and $M''_{\rho,\alpha}$ and obtain sharp upper bound of the Fekete–Szegö functional, $|a_3 - \mu a_2^2|$ over $M'_{\rho,\alpha}$ and $M''_{\rho,\alpha}$. We also give sufficient conditions when they are subsets of $S^*(\alpha)$ and $K(\alpha)$, respectively.

2. Results concerning Fekete–Szegö problem

In this section we give sharp estimates of $|a_2|$ and of the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for a function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ from $M'_{\rho,\alpha}$ and $M''_{\rho,\alpha}$. We will use following lemmas.

Lemma 1. ([5], p.166, formula (10)) ([1], p.41) *Let $p \in \mathcal{P}$, that is, p be analytic in $\mathcal{U}$, be given by $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ and* $\mathrm{Re}\,p(z) > 0$ *for $z \in \mathcal{U}$. Then $|p_2 - p_1^2/2| \le 2 - |p_1|^2/2$ and $|p_n| \le 2$ for all $n \in \mathbf{N}$.*

Lemma 2. *Let $\omega(z) = \sum_{n=1}^{\infty} b_n z^n$ be an analytic function in the unit disk $\mathcal{U}$ and $|\omega(z)| < 1$, $z \in \mathcal{U}$. Then $|b_1| \le 1$ and $|b_2| \le 1 - |b_1|^2$.*

Proof. Define a function $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$ by $p(z) = (1 - \omega(z))/(1 + \omega(z))$. Then $c_1 = -p_1/2$, $c_2 = (p_1^2/2 - p_2)/2$ and the rest follows from Lemma 1. □

The Fekete–Szegö problem for the class $M'_{\rho,\alpha}$ is solved in the following theorem.

Theorem 1. *Let $f \in M'_{\rho,\alpha}$. Then $|a_2| \le \rho/|2 - \alpha|$ and*

$$|a_3 - \mu a_2^2| \le \rho \max \left\{ \frac{1}{|3 - \alpha|}, \frac{\rho|\mu|}{|2 - \alpha|^2} \right\}$$

for all μ in the complex plane. This bounds are sharp.

Proof. From $f \in M'_{\rho,\alpha}$ follows that $f'(z) - \alpha f(z)/z + \alpha - 1 = \rho\omega(z)$, where $\omega(z) = \sum_{n=1}^{\infty} b_n z^n$ is an analytic function such that $|\omega(z)| < 1$, $z \in \mathcal{U}$. So, equating coefficients we obtain $a_2 = b_1\rho/(2-\alpha)$, $a_3 = b_2\rho/(3-\alpha)$ and further, using Lemma 2,

$$|a_3 - \mu a_2^2| = \rho\left[\frac{1}{|3-\alpha|} + x\left(\frac{\rho|\mu|}{|2-\alpha|^2} - \frac{1}{|3-\alpha|}\right)\right] \equiv H(x),$$

where $x = |b_1|^2 \le 1$. Thus, $|a_3 - \mu a_2^2| \le \rho \max\{H(0), H(1)\}$. This inequality is sharp as the functions $f(z) = z + \rho z^2/(2-\alpha)$ and $f(z) = z + \rho z^3/(3-\alpha)$, show.

In the same manner we solve the Fekete–Szegö problem for the class $M''_{\rho,\alpha}$.

Theorem 2. *Let $f \in M''_{\rho,\alpha}$. Then $|a_2| \le \rho/(2|1-\alpha|)$ and*

$$|a_3 - \mu a_2^2| \le \rho \max\left\{\frac{1}{3|2-\alpha|}, \frac{\rho|\mu|}{4|1-\alpha|^2}\right\}$$

for all μ in the complex plane. This bounds are sharp.

3. Results concerning starlikeness and convexity of higher order

In this section we will make use of the well known Jack lemma.

Lemma 3. ([3]) *Let $\omega(z)$ be a non-constant and analytic function in the unit disk $\mathcal{U}$ with $\omega(0) = 0$. If $|\omega(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_0 then $z_0\omega'(z_0) = k\omega(z_0)$ and $k \ge 1$.*

Using Jack lemma we prove the following result.

Theorem 3. *Let $p(z)$ be an analytic function in the unit disk $\mathcal{U}$, $p(0) = 1$ and let $\beta \ne -1$ be a complex number such that* $\mathrm{Re}\beta \ge -1$. *If*

$$|zp'(z) + \beta p(z) - \beta| < \rho \tag{2}$$

for all $z \in \mathcal{U}$ then

$$|p(z) - 1| < \rho/|1+\beta| \tag{3}$$

for all $z \in \mathcal{U}$, and the result is sharp as the function $p(z) = 1 + \rho z/|1+\beta|$ shows.

Proof. For a function $\omega(z)$ defined by $p(z) = 1 + \rho\omega(z)/|1+\beta|$ we can say that it is analytic in $\mathcal{U}$, $\omega(0) = 0$ and

$$zp'(z) + \beta p(z) - \beta = \frac{\rho}{|1+\beta|}\left(z\omega'(z) + \beta\omega(z)\right).$$

Also, inequality (3) is equivalent to $|\omega(z)| < 1$ for all $z \in \mathcal{U}$. Assuming that there exists a $z_0 \in \mathcal{U}$ such that $|\omega(z_0)| = 1$ by the Jack lemma we receive $z_0\omega'(z_0) = k\omega(z_0)$ and $k \geq 1$, and further, $|z_0p'(z_0) + \beta p(z_0) - \beta| = \rho|k+\beta|/|1+\beta| \geq \rho$ because $\mathrm{Re}\beta \geq -1$. This is a contradiction to inequality (2) and so $|\omega(z)| < 1$ for all $z \in \mathcal{U}$. □

Putting $p(z) = f(z)/z$ and $\beta = 1 - \alpha$ in Theorem 3 we obtain next corollary.

Corollary 1. *If* $f \in M'_{\rho,\alpha}$, $\mathrm{Re}\,\alpha \leq 2$ *and* $\alpha \neq 2$ *then* $|f(z)/z - 1| < \rho/|2-\alpha|$, $z \in \mathcal{U}$.

Now we will give sufficient conditions for a function $f \in M'_{\rho,\alpha}$ to be starlike of order α.

Theorem 4. *Let* $f \in M'_{\rho(\alpha),\alpha}$ *with* $\alpha < 1$ *and* $\rho(\alpha) = \frac{(1-\alpha)(2-\alpha)}{\sqrt{(2-\alpha)^2+(1-\alpha)^2}}$. *Then* $\mathrm{Re} zf'(z)/f(z) > \alpha$, $z \in \mathcal{U}$. *Specially, if* $0 \leq \alpha < 1$ *then* $M'_{\rho(\alpha),\alpha} \subset S^*(\alpha)$.

Proof. From $f \in M'_{\rho,\alpha}$ follows that there exists an analytic function $\omega(z)$ such that

$$\frac{zf'(z)}{f(z)} - \alpha = \frac{1-\alpha+\rho\omega(z)}{f(z)/z},$$

$\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathcal{U}$. Since conditions of Corollary 1 are satisfied and $\rho(\alpha) < 1 - \alpha$ we get

$$\begin{aligned}\left|\arg\left(\frac{zf'(z)}{f(z)} - \alpha\right)\right| &\leq \left|\arg(1-\alpha+\rho(\alpha)\omega(z))\right| + \left|\arg\frac{f(z)}{z}\right| \\ &\leq \arcsin\frac{\rho(\alpha)}{|1-\alpha|} + \arcsin\frac{\rho(\alpha)}{|2-\alpha|} \\ &= \arcsin\left(\frac{\rho(\alpha)}{|1-\alpha|}\sqrt{1-\frac{\rho^2(\alpha)}{|2-\alpha|^2}} + \frac{\rho(\alpha)}{|2-\alpha|}\sqrt{1-\frac{\rho^2(\alpha)}{|1-\alpha|^2}}\right) = \frac{\pi}{2},\end{aligned}$$

and so, $\mathrm{Re} zf'(z)/f(z) > \alpha$ for all $z \in \mathcal{U}$. □

Remark 1. *For* $\alpha = 0$ *in Theorem 4 we receive that* $|f'(z)-1| < 2/\sqrt{5}$, $z \in \mathcal{U}$, *implies* $f \in S^*$, *result equivalent to the one from Theorem 2 in [4].*

Imitating the technique from Corollary 1 and Theorem 4 we will obtain conditions when $f \in M''_{\rho(\alpha),\alpha}$ is convex of some order. So, for $p(z) = f'(z)$ and $\beta = -\alpha$ in Theorem 3 we have

Corollary 2. *If* $f \in M''_{\rho(\alpha),\alpha}$, $\mathrm{Re}\alpha \leq 1$ *and* $\alpha \neq 1$ *then* $|f'(z) - 1| < \rho/|1-\alpha|$, $z \in \mathcal{U}$.

Using this corollary we will prove next theorem.

Theorem 5. *Let* $f \in M''_{\rho,\alpha}$, $\alpha < 0$ *and* $\rho(\alpha) = \frac{\alpha(\alpha-1)}{\sqrt{\alpha^2+(1-\alpha)^2}}$. *Then* $\mathrm{Re}\frac{zf''(z)}{f'(z)} > \alpha$, $z \in \mathcal{U}$. *Specially, if* $-1 \le \alpha < 0$ *then* $M''_{\rho(\alpha),\alpha} \subset K(1+\alpha)$.

Proof. Condition $f \in M''_{\rho,\alpha}$ implies existence of an analytic function $\omega(z)$ such that

$$\frac{zf''(z)}{f'(z)} - \alpha = \frac{-\alpha + \rho\omega(z)}{f'(z)},$$

$\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in \mathcal{U}$. Further, from $\rho(\alpha) < -\alpha$ and Corollary 2

$$\begin{aligned}\left|\arg\Big(\frac{zf''(z)}{f'(z)} - \alpha\Big)\right| &\le \big|\arg(-\alpha + \rho(\alpha)\omega(z))\big| + |\arg f'(z)| \\ &\le \arcsin\frac{\rho(\alpha)}{|\alpha|} + \arcsin\frac{\rho(\alpha)}{|1-\alpha|} \\ &= \arcsin\left(\frac{\rho(\alpha)}{|\alpha|}\sqrt{1 - \frac{\rho^2(\alpha)}{|1-\alpha|^2}} + \frac{\rho(\alpha)}{|1-\alpha|}\sqrt{1 - \frac{\rho^2(\alpha)}{|\alpha|^2}}\right) = \frac{\pi}{2}.\end{aligned}$$

Thus $\mathrm{Re} zf''(z)/f'(z) > \alpha$ for all $z \in \mathcal{U}$. □

We can rewrite Theorem 5 in the following, equivalent form.

Theorem 6. *Let* $f \in M''_{\rho,\alpha-1}$ *with* $\alpha < 1$ *and* $\rho(\alpha) = \frac{(1-\alpha)(2-\alpha)}{\sqrt{(1-\alpha)^2+(2-\alpha)^2}}$. *Then* $\mathrm{Re}[1 + zf''(z)/f'(z)] > \alpha$, $z \in \mathcal{U}$. *Specially, if* $0 \le \alpha < 1$ *then* $M''_{\rho(\alpha),\alpha-1} \subset K(\alpha)$.

For $\alpha = 0$ we have next result.

Corollary 3. *Let* $f \in \mathcal{A}$ *and* $|zf''(z) + f'(z) - 1| < 2\sqrt{5}$ *for all* $z \in \mathcal{U}$. *Then* $f \in K$.

References

[1] P. L. Duren: *"Univalent functions,"* Springer-Verlag, New-York, 1983.

[2] R. Fournier and P. Mocanu: *Differential inequalities which imply starlikeness,* preprint.

[3] I. S. Jack: *Functions starlike and convex of order* α, J. London Math. Soc. **2** 3 (1971), 469–474.

[4] P. Mocanu: *Some starlikeness conditions for analytic functions,* Rev. Roumaine Math. Pures. Appl. 33(1988), 117-124.

[5] Ch. Pommerenke: *"Univalent Functions,"* Vandenhoeck & Ruprecht, Göttingen, 1975.

[6] S. Ponnusamy and V. Singh: *Criteria for univalent, starlike and convex functions,* preprint.

ЗА ДВЕ КЛАСИ АНАЛИТИЧКИ ФУНКЦИИ

Никола Тунески

Резиме

Нека $\mathcal{A}$ е класата аналитички функции дефинирани на единечниот диск $\mathcal{U} = \{z : |z| < 1\}$ нормализирани така што $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Класите

$$M'_{\alpha,\rho} = \{f \in \mathcal{A} : |f'(z) - \alpha f(z)/z + \alpha - 1| < \rho, z \in \mathcal{U}\}$$

и

$$M''_{\alpha,\rho} = \{f \in \mathcal{A} : |zf''(z) - \alpha f'(z) + \alpha| < \rho, z \in \mathcal{U}\}$$

се претходно проучувани од Fournier–Mocanu и Ponnusamy–Singh. Во овој труд добиена е најдобра горна граница на Fekete–Szegö функционалот врз овие две класи. Исто така, дадени се доволни услови кои ги сместуваат овие класи во класата на ѕвездолики функции од ред α и во класата конвексни функции од ред α.

St. Cyril and Methodius University

Faculty of Mechanical Engineering

Karpoš II b.b.

1000 Skopje

Republic of Macedonia

e-mail: nikolat@mf.ukim.edu.mk