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TESTING ALZER'S INEQUALITY FOR MATHIEU SERIES $S(r)$

BISERKA DRAŠČIĆ AND TIBOR K. POGÁNY

Abstract. Consider the Mathieu series $S(r) = \sum_{n=1}^{\infty} 2n(n^2+r^2)^{-2}$. We interpolate the Alzer's bilateral bounding inequality in the following manner. We find intervals I_1, I_2 such that

$$\frac{1}{r^2+\kappa_1} \le 2\int_1^{\infty} \frac{[\sqrt{t}]^2}{(r^2+t)^3}\,dt \le S(r), \qquad r \in I_1$$

$$S(r) < 4\int_1^{\infty} \frac{[\sqrt{t}]}{(r^2+t)^3}\,dt + 2\int_1^{\infty} \frac{[\sqrt{t}]^2}{(r^2+t)^3}\,dt \le \frac{1}{r^2+\kappa_2} \qquad r \in I_2.$$

Here $\kappa_1 = 1/(2\zeta(3)), \kappa_2 = 1/6$.

1. Introduction

The long history and preliminaries of Mathieu and aligned inequalities we can follow by reading [1], [5]. In [3] Mathieu is defined

$$S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^2}, \qquad r > 0,$$

such that we call *Mathieu series*. In their article Alzer *et al.* proved that the following bilateral inequality is sharp:

$$\frac{1}{r^2+\kappa_1} < S(r) < \frac{1}{r^2+\kappa_2}, \tag{1}$$

where

$$\kappa_1 = \frac{1}{2\zeta(3)}, \quad \kappa_2 = \frac{1}{6}.$$

In [2],[6],[7, Open Problem 4.3] the generalized Mathieu series is introduced:

$$S_p(r,\alpha) = \sum_{n=1}^{\infty} \frac{2n^{\alpha/2}}{(n^\alpha+r^2)^p}, \qquad S_2(r,2) \equiv S(r), \qquad r, p+1, \alpha > 0.$$

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In [4] the second author gives the following inequality:

$$0 \le S_p(r,\alpha) - \frac{4(p+1)}{\alpha+2}\int_1^\infty \frac{[t^{1/\alpha}]^{\alpha/2+1}}{(r^2+t)^{p+2}}\,dt < 2(p+1)\int_1^\infty \frac{[t^{1/\alpha}]^{\alpha/2}}{(r^2+t)^{p+2}}\,dt, \tag{2}$$

which is sharp in sense of sharpness of $0 \le \{z\} < 1$. (Here $[z], \{z\}$ denotes the integer and the fractional part of $z \in \mathbb{R}$.)

Putting $p = 1, \alpha = 2$ in (2) we get

$$L_r \le S(r) < L_r + D_r, \tag{3}$$

where

$$L_r := 2\int_1^\infty \frac{[\sqrt{t}]^2}{(r^2+t)^3}\,dt; \qquad D_r := 4\int_1^\infty \frac{[\sqrt{t}]}{(r^2+t)^3}\,dt\,. \tag{4}$$

Comparing (1) and (3) it is obvious that for certain values of r has to be

$$\frac{1}{r^2+\kappa_1} \le L_r \le S(r), \tag{5}$$

since (3) allows $\le$ in lower bound [1]. So we are looking for those $I_1 \in \mathbb{R}^+$ which confirms (5) for all $r \in I_1$. Similar question arises immediately for the case of upper bounds, namely we will ask for certain I_2 such that

$$S(r) < L_r + D_r \le \frac{1}{r^2+\kappa_2}, \qquad r \in I_2. \tag{6}$$

Both results will be synthetized into Theorem in the next chapter.

2. Main results

Consider the function

$$\varphi(r) = L_r - \frac{1}{r^2+\kappa_1}, \qquad r > 0. \tag{7}$$

When $\varphi(r) \ge 0$, then for those r the bound L_r is better than Alzer's one.

Theorem 1. *The inequality*

$$\frac{1}{r^2+\kappa_1} \le L_r \le S(r) \tag{8}$$

holds for all $r \in I_1 = [r_1, r_2]$, *where* r_1, r_2 *are the real positive roots of the equation*

$$\frac{r^2+3}{(r^2+1)^2} - \frac{8}{3(r+1)^3} + \frac{4r}{(r+1)^4} - \frac{8r^2}{5(r+1)^5} = \frac{1}{r^2+\kappa_1},$$

where (11) *we consider according to* footnote 1. *Moreover, the inequality*

$$S(r) < L_r + D_r \le \frac{1}{r^2+\kappa_2} \tag{9}$$

[1]**Remark 1.** At this point we note that in (5) both equalities cannot happen simultaneously.

holds for all $r \in I_2 = \langle 0, r_3] \cup [r_4, \infty\rangle$, *where* r_3, r_4 *are the positive real roots of the equation*

$$\frac{\pi r^4 + 2(4+\pi)r^2 + \pi - 4}{4r^2(1+r^2)^2} - \frac{1}{r^2+\kappa_2} = 0.$$

In (8),(9) *for* $r_\ell, \ell = \overline{1,4}$ *we have equalities.*

Proof. As L_r is not easily handable, we will minorize L_r. Because $[\sqrt{t}]^2 \geq (\sqrt{t}-1)^2$, it follows

$$L_r \geq 2\int_1^\infty \frac{(\sqrt{t}-1)^2}{(r^2+t)^3}\,dt = \frac{r^2+3}{(r^2+1)^2} - 8\int_1^\infty \frac{u^2}{(r^2+u^2)^3}\,du\,.$$

Finally, we arrive at

$$\varphi(r) \geq \frac{r^2+3}{(r^2+1)^2} - \frac{8}{3(r+1)^3} + \frac{4r}{(r+1)^4} - \frac{8r^2}{5(r+1)^5} - \frac{1}{r^2+\kappa_1}. \tag{10}$$

To find I_1, it is enough to solve

$$\frac{r^2+3}{(r^2+1)^2} - \frac{8}{3(r+1)^3} + \frac{4r}{(r+1)^4} - \frac{8r^2}{5(r+1)^5} - \frac{1}{r^2+\kappa_1} = 0. \tag{11}$$

Using *Mathematica 5.0* we get two real roots of (11):

$$r_1 \approx 0.394443, \qquad r_2 \approx 5.04572.$$

By the same tool we test all other characteristics of the function

$$f(r) = \frac{r^2+3}{(r^2+1)^2} - \frac{8}{3(r+1)^3} + \frac{4r}{(r+1)^4} - \frac{8r^2}{5(r+1)^5} - \frac{1}{r^2+\kappa_1}. \tag{12}$$

As $f(r)$ attains its maximum at $r \approx 0.716248$ and it is minimal at $r \approx 6.80008$. According to this, we can see that between the roots r_1 and r_2 function is positive, i.e.

$$\varphi(r) \geq f(r) \geq 0, \quad r \in I_1 = [r_1, r_2].$$

It only remains to show that $f(r) < 0$ for all $r > r_2$. Namely, it is not enough to show that no other real zeros are there in $f(r) = 0$, as $r \to \infty$. So in this purpose let us transform $f(r)$

$$f(r) \leq \frac{\kappa_1 - 1}{(r^2+1)(r^2+\kappa_1)} - \frac{4r^2 - 100r - 80}{15(r+1)^5}.$$

The right side of inequality is negative when $4r^2 - 100r - 80 > 0$, and this is true for $r > r_5 \approx 25.75$. So $f(r) < 0$ for all $r > r_5$. The critical interval is $\langle r_2, r_5\rangle$. Because $f(r)$ has it's minimal value for $r_6 \approx 6.80008 \in \langle r_2, r_5\rangle$, we split $\langle r_2, r_5\rangle$ into two subintervals $\langle r_2, r_6\rangle$ and $\langle r_6, r_5\rangle$, say. As $f'(r) < 0$ in the first interval, $f'(r) > 0$ on the second interval and $f(25.75) < 0$, we finish the prof of the left hand inequality (8) for all $r \in I_1$.

Now, we prove the inequality (9). It follows

$$S(r) < L_r + D_r \le 2\int_1^\infty \frac{t+2\sqrt{t}}{(r^2+t)^3}\,dt = \frac{\pi r^4 + 2(4+\pi)r^2 + \pi - 4}{4r^2(1+r^2)^2}.$$

We define

$$\varphi_1(r) = \frac{\pi r^4 + 2(4+\pi)r^2 + \pi - 4}{4r^2(1+r^2)^2} - \frac{1}{r^2+\kappa_2}.$$

Using *Mathematica 5.0* we find that $\varphi_1(r) \le 0$ holds for $r \in \langle 0, r_3\rangle \cup \langle r_4, \infty\rangle$, where

$$r_3 \approx 0.660463, \qquad r_4 \approx 2.74663.$$

It is obvious that

$$\varphi_1(r) \sim \Big(\frac{\pi}{4} - 1\Big) r^{-2}, \qquad r \to \infty,$$

therefore $\varphi_1(r) < 0$, $r \in I_2$.

This finishes the proof of theorem. □

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Department of Sciences, Faculty of Maritime Studies, University of Rijeka, 51000 Rijeka, Studentska 2, Croatia

E-mail address: `bdrascic@pfri.hr, poganj@pfri.hr`