ON UNIFORM CONVERGENCE OF FOURIER SERIES

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SUMMARY. — A theorem on uniform convergence of Fourier series containing the well-known Dini-Lipschitz test for uniform convergence and a theorem of M. Tomić, is given here.

1. (i) Let C denote the class of continuous and periodic functions of period 2π and C_F the class of functions $f(x) \in C$ having uniformly convergent Fourier series.

The modulus of continuity of $f(x) \in C$ is as usually defined by

$$\omega(t) = \operatorname*{Max}_{|x-y| \leqq t} |f(x) - f(y)|.$$

Let $\Omega(t)$ be a continuous function decreasing monotonically to 0 as $t \to +0$, and

$$\Omega(0) = 0, \qquad \Omega(x+y) \leqslant \Omega(x) + \Omega(y).$$

Then there exists a function $f(x) \in C$ having the modulus of continuity $\Omega(t)$ ([9], p. 486).

A function $f(x) \in C$ belongs to the class C^{Ω} if its modulus of continuity $\omega(t)$ satisfies the condition

$$\omega(t) = O\{\Omega(t)\}, \quad t \to 0.$$

Given two classes P and Q of integrable functions, we shall denote, following Zygmund ([3], p. 100), by (P, Q) the class of sequences $\{\lambda_n\}$ such that

(1) $$\frac{1}{2}\lambda_0 a_0 + \sum_{\nu=1}^{\infty} \lambda_\nu (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

is the Fourier series of a function $f^*(x) \in Q$ whenever

(2) $$\frac{1}{2} a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

is the Fourier series of a function $f(x) \in P$.

(ii) Recently J. Karamata ([1]; [2], p. 127) proved the following theorem:

A necessary and sufficient condition that $\{\lambda_n\}$ *should belong to the class* (C, C_F) *is*

$$\int_0^{2\pi} |\Lambda_n(t)|\, dt = O(1), \tag{3}$$

where

$$\Lambda_n(t) = \frac{1}{2}\lambda_0 + \sum_{\nu=1}^{n} \lambda_\nu \cos \nu t.$$

In other words, the condition (3) is necessary and sufficient in order that the series (1) should be uniformly convergent for an arbitrary $f(x) \in C$.

This theorem contains in particular the theorem ([3], p. 58—59, [6], p. 23)

If $\{\lambda_n\}$ *is quasi-convex, i.e. if*

$$\sum_{\nu=0}^{\infty} (\nu+1)|\Delta^2 \lambda_\nu| < \infty, \text{ where } \Delta^2\lambda_\nu = \lambda_\nu - 2\lambda_{\nu+1} + \lambda_{\nu+2},$$

and $\lambda_n \lg n = O(1)$, *then* $\{\lambda_n\}$ *belongs to the class* (C, C_F).

The condition (3) imposes a restriction on the behavior of λ_n which is very severe. H. Helson [7] has shown that from (3) it follows $\lambda_n = o(1)$.

However, if we consider the class C^Ω, instead of the class C, the condition (3) can be essentially enlarged. Here we shall prove the following

THEOREM. *If*

$$\int_0^{2\pi} \left| \sum_{\nu=0}^{n} \Lambda_\nu(t) \right| dt = O(n) \tag{4}$$

and

$$\Omega\left(\frac{1}{n}\right) \int_0^{2\pi} |\Lambda_n(t)|\, dt = o(1), \tag{5}$$

then $\{\lambda_n\}$ *belongs to te class* (C^Ω, C_F).

In other words, if $\{\lambda_n\}$ satisfies conditions (4) and (5), then the series (1) is uniformly convergent for an arbitrary $f(x) \in C^\Omega$.

Our theorem may be regarded as a generalization of the well known Dini—Lipschitz test for uniform convergence ([3], p. 30), to which it reduces when $\lambda_n = 1$, $n = 0, 1, 2, \ldots$ It contains also the following theorem proved first by M. Tomić ([6], p. 24, [8])

If $\{\lambda_n\}$ *is quasi-convex and* $\Omega(1/n)\,\lambda_n \lg n = o(1)$, *then* $\{\lambda_n\}$ *belongs to the class* (C^{Ω}, C_F).

2. (i) The proof of the just cited theorem, which will be given here, depends on Jackson's fundamental theorem on the approximation of a continuous function by trigonometrical polynomials ([4], p. 7). We shall use Jackson's theorem in the following form ([5], p. 319):

If $f(x) \in C$ *and*

$$T_n(x) = \sum_{\nu=0}^{2n-1} D(\nu/n)\,(a_\nu \cos \nu x + b_\nu \sin \nu x), \tag{6}$$

where

$$D(x) = \begin{cases} 1 - {}^3/_2\, x^2 + {}^3/_4\, x^3, & 0 \leqslant x \leqslant 1, \\ {}^1/_4\,(2-x)^3, & 1 \leqslant x \leqslant 2, \\ 0, & x \geqslant 2, \end{cases} \tag{7}$$

then

$$|f(x) - T_n(x)| \leqslant A\omega(1/n). \tag{8}$$

This problem was suggested by M. Tomić to which we are very indebted for his constant advice and criticism.

(ii) Proof of the Theorem. We note first that (4) is a necessary and sufficient condition that the series

$$\frac{1}{2}\lambda_0 + \sum_{\nu=1}^{\infty} \lambda_\nu \cos \nu t \tag{9}$$

should be a Fourier-Stieltjes series ([3], p. 79). On the other hand, a necessary and sufficient condition for $\{\lambda_n\}$ to belong to the class (C, C) is that the series (9) should be a (Fourier-Stieltjes series ([3], p. 101).

From these facts it follows that (4) is a necessary and sufficient condition for $\{\lambda_n\}$ to belong to the class (C, C).

Since $f(x) \in C$, it follows therefore from (4) that (1) is the Fourier series of a continuous function $f^*(x) \in C$.

Let $s_n^*(x)$ be the n-th partial sum of the series (1) and let $\sigma_n^*(x)$ be the first arithmetic mean of $s_n^*(x)$. We have

$$|f^*(x) - s_n^*(x)| \leqslant |f^*(x) - \sigma_n^*(x)| + |\sigma_n^*(x) - s_n^*(x)|.$$

Now, since $f^*(x)$ is continuous, $\sigma_n^*(x)$ converges uniformly to $f^*(x)$ in virtue of Fejér's theorem ([3], p. 45), and it remains to be proved that

$$\sigma_n^*(x) - s_n^*(x) = \frac{1}{n+1}\sum_{\nu=1}^{n} \nu\lambda_\nu (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

converges uniformly to 0 as $n \to \infty$.

Let $T_n(x)$ be the Jackson's polynomial corresponding to the function $f(x) \in C^\Omega$. Then

$$s_n^*(x) = \frac{1}{\pi}\int_0^{2\pi} f(x+t)\,\Lambda_n(t)\,dt =$$

$$= \frac{1}{\pi}\int_0^{2\pi} T_n(x+t)\,\Lambda_n(t)\,dt + \frac{1}{\pi}\int_0^{2\pi} \{f(x+t) - T_n(x+t)\}\,\Lambda_n(t)\,dt.$$

Let us denote the last term on the right by $A_n(x)$. Then from (6) it follows that

$$(10)\qquad s_n^*(x) = \frac{1}{2}\lambda_0 a_0 + \sum_{\nu=1}^{n} D\left(\frac{\nu}{n}\right)\lambda_\nu (a_\nu \cos \nu x + b_\nu \sin \nu x) + A_n(x).$$

Since, according to (7),

$$D\left(\frac{\nu}{n}\right) = 1 - \frac{3}{2}\left(\frac{\nu}{n}\right)^2 + \frac{3}{4}\left(\frac{\nu}{n}\right)^3,\quad 0 \leqslant \nu \leqslant n.$$

we obtain from (10)

$$(11)\qquad 2n\sum_{\nu=1}^{n} \nu^2 \alpha_\nu(x) - \sum_{\nu=1}^{n} \nu^3 \alpha_\nu(x) = {}^4/_3\, n^3 A_n(x),$$

where $\alpha_\nu(x) = \lambda_\nu (a_\nu \cos \nu x + b_\nu \sin \nu x)$.

Now, if $\Delta^2 w_n = w_n - 2w_{n+1} + w_{n+2}$, then

$$\Delta^2\left\{n\sum_{\nu=1}^{n} u_\nu\right\} = (n+2)\,u_{n+2} - n u_{n+1}$$

and

$$\Delta^2\left\{\sum_{\nu=1}^{n} v_\nu\right\} = v_{n+2} - v_{n+1}.$$

Applying Δ^2 to the equation (11) we obtain easily

$$(n+2)^3\alpha_{n+2}(x) - (n+1)^2(n-1)\,\alpha_{n+1}(x) = {}^4/_3\,\Delta^2\{n^3 A_n(x)\}.$$

Multiplying this equation by $n(n+1)$ and summing we obtain

$$^3/_4\, n^3(n-1)(n-2)\,\alpha_n(x)=\sum_{\nu=1}^{n-2}\nu(\nu+1)\,\Delta^2\{\nu^3 A_\nu(x)\}=$$

$$=n^3(n-1)(n-2)\,A_n(x)-(n+1)(n-1)^3(n-2)\,A_{n+1}(x)+2\sum_{\nu=1}^{n-2}\nu^3 A_\nu(x),$$

or

$$^3/_4\, n\,\alpha_n(x)=n\,A_n(x)-(n-1)\,A_{n-1}(x)+\frac{n-1}{n^2}A_{n-1}(x)+$$

$$+\frac{2}{n^2(n-1)(n-2)}\sum_{\nu=1}^{n-2}\nu^3 A_\nu(x).$$

Summing once again we get

$$^3/_4\sum_{\nu=1}^{n}\nu\,\alpha_\nu(x)=n\,A_n(x)+\sum_{\nu=1}^{n-1}\frac{\nu}{(\nu+1)^2}A_\nu(x)+$$

$$+2\sum_{\nu=1}^{n-2}\frac{1}{(\nu+2)^2(\nu+1)\,\nu}\sum_{k=1}^{n}k^3 A_k(x),$$

and so

(12) $$^3/_4\left|\sum_{\nu=1}^{n}\nu\,\alpha_\nu(x)\right|\leqslant n\,|A_n(x)|+\sum_{\nu=1}^{n}|A_\nu(x)+2\sum_{\nu=1}^{n}\nu^{-4}\sum_{k=1}^{\nu}k^3\,|A_k(x)|.$$

Since, for $k\geqslant 1$

$$\sum_{\nu=k}^{\infty}\nu^{-4}=k^{-4}+\int_k^\infty x^{-4}\,dx<2\,k^{-3}\quad .$$

we have

$$\sum_{\nu=1}^{n}\nu^{-4}\sum_{\nu=1}^{\nu}k^3\,|A_k(x)|=\sum_{k=1}^{n}k^3\,|A_k(x)|\sum_{\nu=k}^{\infty}\nu^{-4}\leqslant$$

$$\leqslant 2\sum_{k=1}^{n}|A_k(x)|,$$

so that the inequality (12) reduces to

$$\frac{1}{n+1}\left|\sum_{\nu=1}^{n}\nu\,\alpha_\nu(x)\right|\leqslant 2\,|A_n(x)|+\frac{10}{n+1}\sum_{\nu=1}^{n}|A_\nu(x)|.$$

Hence, it remains to be proved that $A_n(x)$ converges uniformly to 0 as $n \to \infty$. For $n \geqslant 1$ we have, according to (9)

$$|A_n(x)| = \frac{1}{\pi} \int_0^{2\pi} \left| \{ f(x+t) - T_n(x+t) \} \Lambda_n(t)\, dt \right| \leqslant$$

$$\leqslant \frac{1}{\pi} \int_0^{2\pi} | f(x+t) - T_n(x+t) |\, |\Lambda_n(t)|\, dt \leqslant$$

$$\leqslant A\, \omega\left(\frac{1}{n}\right) \int_0^{2\pi} |\Lambda_n(t)|\, dt,$$

and so for an arbitrary $f(x) \in C^{\Omega}$, according to (5),

$$|A_n(x)| \leqslant B\, \Omega\left(\frac{1}{n}\right) \int_0^{2\pi} |\Lambda_n(t)|\, dt \to 0, \quad n \to \infty.$$

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