ON COMPLETELY SIMPLE SEMIGROUPS

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A semigroup S is called completely simple without zero[1], if it contains a minimal left ideal and a minimal right one, and if it does not contain proper two-sided ideals.

Let: (*i*) G be a group; (*ii*) L and R be two non-empty sets, (*iii*) λ be a mapping of $R \times L$ in G, (*iv*) $S = G \times L \times R$, and (*v*) a product of two elements of S be defined by

$$(g_1 ; l_1, r_1)(g_2 ; l_2, r_2) = (g_1 \lambda (r_1, l_2) g_2 ; l_1, r_2) \quad (g_i \in G, l_i \in L, r_i \in R).$$

Then $S = S(G ; L, R ; \lambda)$ is a completely simple semigroup. If $A = L \times R$ and $G_a = \{(x ; a), x \in G\}$, then $F = \{G_\delta ; \delta \in A\}$ is the collection of all maximal subgroups of S, and the following two equations are also true:

$$S = \bigcup_{\delta \in A} G_\delta , \tag{1}$$

$$G_\delta S G_\delta = G_\delta , \text{ for every } \delta \in A . \tag{2}$$

It is well known (see, for example [2] p. 500 or [3] p. 291) that each completely simple semigroup (without zero) is isomorphic with a semigroup $S(G ; L, R ; \lambda)$. Therefore, if $F = \{G_\delta ; \delta \in A\}$ is the collection of all maximal subgroups of the completey simple semigroup S, then the statements (1) and (2) are valid.

The purpose of this paper is to prove the following results.*

T h e o r e m. *Let S be a semigroup such that there exists a collection $F = \{G_\delta ; \delta \in A\}$ of subgroups of S which satisfy* (1) *and* (2). *Then the semigroup S is completely simple, and F is the collection of all maximal subgroups of S.*

C o r o l l a r y 1. *A completely simple semigroup S is isomorphic with the direct product of a group G and an anticommutative semigroup*[2] *A if and only if the set B of all idempotents of S is a subsemigroup of S; then the semigroups A and B are isomorphic.*

C o r o l l a r y 2. *If the semigroup S is a union of subgroups each of which is a left ideal of S, then there exist a group G and a set A such that S is isomorphic with the semigroup $G \times A$, where $(x, \alpha)(x, \beta) = (x, \beta)$.*

First, in Sections **1** and **2**, we prove some more general lemmas, and then, in Section **3**, we prove the Theorem and the Corollaries.

1. Let G_α, G_β and G_γ be subgroups of the semigroup S such that the statements $G_\alpha G_\beta \cap G_\gamma \neq 0$ and (2) hold, for $\delta = \alpha, \beta, \gamma$.

L e m m a 1.1. *Let $\varepsilon, \delta = \alpha, \beta, \gamma$. If $G_\varepsilon \cap G_\delta \neq 0$, then $G_\varepsilon = G_\delta$.*

P r o o f. Let $x \in G_\varepsilon \cap G_\delta$. Then $G_\varepsilon = x G_\varepsilon x \subseteq G_\delta S G_\delta = G_\delta$. Analogously, $G_\delta \subseteq G_\varepsilon$.

L e m m a 1.2. *If $a_\alpha \in G_\alpha$, $a_\beta \in G_\beta$ and $a_\alpha a_\beta \in G_\gamma$, then $G_\gamma \subseteq G_\alpha a_\beta \cap a_\alpha G_\beta$.*

P r o o f. We have $G_\gamma = a_\alpha a_\beta G_\gamma a_\alpha a_\beta \subseteq a_\alpha a_\beta S a_\beta \subseteq a_\alpha G_\beta$. Analogously, $G_\gamma \subseteq G_\alpha a_\beta$.

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[1] Henceforth, the phrase »without zero« will be omitted.

* *Note added in proof.* Most of results of this paper are to be found in the book A. H. C l i f f o r d and G. B. P r e s t o n, The Algebraic Theory of Semigroups, Vol. I, Mathematical Surveys № 7, Amer. Math. Soc. 1961. (Ex. 14 and 15, p. 84 § 2.7, Ex. 2 (b), p. 97 § 3.2), which was not available to the author at the time he submitted the paper for publication.

[2] A semigroup A is anticommutative if $\alpha\beta = \beta\alpha$ implies $\alpha = \beta$, $\alpha, \beta \in A$.

Corollary 2. *If the semigroup S is a union of subgroups each of which is a left ideal of S, then there exist a group G and a set A such that S is isomorphic with the semigroup $G \times A$, where* $(x, \alpha)(x, \beta) = (x, \beta)$.

First, in Sections **1** and **2**, we prove some more general lemmas, and then, in Section **3**, we prove the Theorem and the Corollaries.

1. Let G_α, G_β and G_γ be subgroups of the semigroup S such that the statements $G_\alpha G_\beta \cap G_\gamma \neq 0$ and (2) hold, for $\delta = \alpha, \beta, \gamma$.

Lemma 1.1. *Let* $\varepsilon, \delta = \alpha, \beta, \gamma$. *If* $G_\varepsilon \cap G_\delta \neq 0$, *then* $G_\varepsilon = G_\delta$.

Proof. Let $x \in G_\varepsilon \cap G_\delta$. Then $G_\varepsilon = x\, G_\varepsilon\, x \subseteq G_\delta\, S\, G_\delta = G_\delta$. Analogously, $G_\delta \subseteq G_\varepsilon$.

Lemma 1.2. *If* $a_\alpha \in G_\alpha$, $a_\beta \in G_\beta$ *and* $a_\alpha a_\beta \in G_\gamma$, *then* $G_\gamma \subseteq \subseteq G_\alpha a_\beta \cap a_\alpha G_\beta$.

Proof. We have $G_\gamma = a_\alpha a_\beta\, G_\gamma\, a_\alpha a_\beta \subseteq a_\alpha a_\beta\, S\, a_\beta \subseteq a_\alpha G_\beta$. Analogously, $G_\gamma \subseteq G_\alpha a_\beta$.

Proof. Let x_α be an arbitrary element of G_α. By Lemma 1.5, $G_\gamma = x_\alpha a_\alpha^{-1} G_\gamma$, whence (by 1.2) $G_\gamma = x_\alpha a_\alpha^{-1} G_\gamma \subseteq x_\alpha a_\alpha^{-1} a_\alpha G_\beta = = x_\alpha G_\beta$. Similarly, $G_\gamma \subseteq G_\alpha x_\beta$.

Lemma 1.7. *If* e_δ *is the neutral element of* G_δ *then*

$$e_\alpha = e_\gamma e_\alpha, \quad e_\beta = e_\beta e_\gamma, \quad e_\gamma = e_\alpha e_\gamma = e_\gamma e_\beta.$$

Proof. By 1.3, $e_\gamma e_\alpha \in G_\alpha$. If x_α is an arbitrary element of G_α, then (by 1.3) there is an element b_γ of G_γ so that $x_\alpha = b_\gamma e_\alpha$. Hence, $x_\alpha = b_\gamma e_\alpha = b_\gamma e_\gamma e_\alpha = b_\gamma e_\alpha e_\gamma e_\alpha = x_\alpha e_\gamma e_\alpha$, i. e. $e_\gamma e_\alpha = e_\alpha$. Similarly, $e_\beta = e_\beta e_\gamma$. It is clear that $e_\gamma e_\alpha = e_\alpha$ and $e_\beta e_\gamma = e_\beta$ (according to the above lemmas) imply $e_\gamma = e_\alpha e_\gamma = e_\gamma e_\beta$.

Lemma 1.8. *The subgroups* G_α, G_β *and* G_γ *are isomorphic.*

Proof. Let $x_\alpha \varphi_{\alpha\gamma} = x_\alpha e_\gamma$ and $x_\gamma \varphi_{\gamma\alpha} = x_\gamma e_\alpha$. By 1.7, we have $x_\alpha \varphi_{\alpha\gamma} \varphi_{\gamma\alpha} = x_\alpha$ and $x_\gamma \varphi_{\gamma\alpha} \varphi_{\alpha\gamma} = x_\gamma$, i. e. $\varphi_{\alpha\gamma}$ is an one-to-one mapping G_α onto G_γ and $\varphi_{\gamma\alpha} = \varphi_{\alpha\gamma}^{-1}$. We have also

$$(x_\alpha y_\alpha) \varphi_{\alpha\gamma} = x_\alpha y_\alpha e_\gamma = x_\alpha e_\gamma y_\alpha e_\gamma = x_\alpha \varphi_{\alpha\gamma} \cdot y_\alpha \varphi_{\alpha\gamma},$$

i. e. $\varphi_{\alpha\gamma}$ is an isomorphism of G_α on G_γ. If we put $x_\beta \psi_{\beta\gamma} = e_\gamma x_\beta$ and $x_\gamma \psi_{\gamma\beta} = e_\beta x_\gamma$; in a similar way it can be shown that $\psi_{\beta\gamma}$ is an isomorphism of G_β on G_γ and $\psi_{\gamma\beta} = \psi_{\beta\gamma}^{-1}$.

2. Let $F = \{G_\delta;\ \delta \in A\}$ be a system of subgroups of the semigroup S, such that the statement (2) holds.

From the Lemmas 1.6 and 1.8 it follows:

Lemma 2.1. *Let* $\alpha, \beta \in A$ *and let* $\gamma \in A' \Leftrightarrow G_\alpha G_\beta \cap G_\gamma \neq 0$. *Then* $\bigcup_{\gamma \in A'} G_\gamma \subseteq x_\alpha G_\beta \cap G_\alpha x_\beta$. If $A' \neq 0$, then all groups of the system $\{G_\delta;\ \delta \in A' \cup \{\alpha, \beta\}\}$ *are isomorphic.*

Lemma 2.2. *If* $G_\alpha G_\beta \subseteq \bigcup_{\delta \in A} G_\delta$, *then there is a* $G_\gamma \in F$ *such that* $G_\alpha G_\beta = x_\alpha G_\beta = G_\alpha x_\beta = G_\gamma$, *for every* $x_\alpha \in G_\alpha$, $x_\beta \in G_\beta$.

Proof. Let $A' (\subseteq A)$ be defined as in Lemma 2.1. Then $G_\alpha G_\beta \subseteq \bigcup_{\delta \in A} G_\delta$ implies $G_\alpha G_\beta \subseteq \bigcup_{\delta \in A'} G_\delta$, whence (by 1.6) we obtain

$$(i) \quad \bigcup_{\delta \in A'} G_\delta = x_\alpha G_\beta = G_\alpha x_\beta = G_\alpha G_\beta.$$

Let G_γ and G_ε be two members of the system $\{G_\delta;\ \delta \in A'\}$, and let $x_\alpha, x_\beta, x_\gamma$ and x_ε be arbitrary elements of $G_\alpha, G_\beta, G_\gamma$ and G_ε respectively. Then, by (*i*), there are elements $y_\alpha, z_\alpha \in G_\alpha$ and $y_\beta, z_\beta \in G_\beta$ such that (*ii*) $x_\gamma = y_\alpha x_\beta = x_\alpha y_\beta$, and (*iii*) $x_\varepsilon = z_\alpha x_\beta = x_\alpha z_\beta$. From (*ii*) and (*iii*) it follows:

[3] x_δ is an arbitrary element of G_δ.

$$(iv) \quad x_\gamma x_\varepsilon = x_\gamma z_\alpha x_\beta \in x_\gamma G_\alpha x_\beta = x_\gamma G_\alpha y_\alpha x_\beta = x_\gamma G_\alpha x_\gamma = G_\gamma$$

and

$$(v) \quad x_\gamma x_\varepsilon = x_\alpha y_\beta x_\varepsilon \in x_\alpha G_\beta x_\varepsilon = x_\alpha z_\beta G_\beta x_\varepsilon = x_\varepsilon G_\beta x_\varepsilon = G_\varepsilon ,$$

i. e.

$$(vi) \quad x_\gamma x_\varepsilon \in G_\gamma \cap G_\varepsilon .$$

From (vi), by 1.1, it follows $G_\varepsilon = G_\gamma$. Therefore, we have $A' = \{\gamma\}$ and this proves our lemma.

3. Proof of the Theorem. Let $F = \{G_\delta ; \delta \in A\}$ be a collection of (different) subgroups of the semigroup S such that the statements (1) and (2) hold.

If we put $\alpha \beta = \gamma \Leftrightarrow G_\alpha G_\beta = G_\gamma$, then (by 2.2) A becomes a semigroup and then we have $G_\alpha G_\beta = x_\alpha G_\beta = G_\alpha x_\beta = G_{\alpha\beta}$, for every $x_\alpha \in G_\alpha$, $x_\beta \in G_\beta$. We have also $G_{\alpha\beta\alpha} = G_\alpha G_\beta G_\alpha \subseteq G_\alpha$, i. e. $\alpha\beta\alpha = \alpha$. Hence, A is an anticommutative semigroup ([3] p. 109).

Let x be an arbitrary element of S and let $x \in G_\alpha$. Then we have

$$S x S = \bigcup_{\beta,\gamma \in A} G_\beta x G_\gamma = \bigcup_{\beta,\gamma \in A} G_{\beta\alpha\gamma} = \bigcup_{\beta,\gamma \in A} G_{\beta\gamma} \supseteq \bigcup_{\beta \in A} G_\beta = S ,$$

because $\beta \alpha \gamma = \beta \gamma$ ([3] p. 109). Therefore, there are no proper two-sided ideals in the semigroup S.

We shall now prove that every left principal ideal is a left minimal one. Let $x_\alpha \in G_\alpha$ and $y = y_{\beta\alpha} \in G_{\beta\alpha} = G_\beta x_\alpha \subseteq S x_\alpha$. Then we have

$$S x_\alpha = \bigcup_{\delta \in A} G_\delta x_\alpha = \bigcup_{\delta \in A} G_{\delta\alpha} = \bigcup_{\delta \in A} G_{\delta\beta\alpha} = \bigcup_{\delta \in A} G_\delta y_{\beta\alpha} =$$
$$= (\bigcup_{\delta \in A} G \delta) y_{\beta\alpha} = S y_{\beta\alpha} .$$

Hence, $S x$ is a minimal left ideal.

In a similar way, it can be shown that every right principal ideal is a right minimal one.

This proves the Theorem.

Proof of Corollary 1. Let G be a group and A an anticommutative semigroup. The direct product $G \times A$ becomes a semigroup if the product is defined by $(x, \alpha)(y, \beta) = (x y, \alpha \beta)$, when $x, y \in G$ and $\alpha, \beta \in A$. If $G_\delta = \{(x, \delta) ; x \in G\}$ then $\{G_\delta ; \delta \in A\}$ is a collection of subgroups of $S = G \times A$, such that the equations (1) and (2) hold. Therefore $G \times A$ is a completely simple semigroup. Then we have $(e, \alpha)(e, \beta) = (e, \alpha\beta)$ (if e is the neutral element of G), i. e. the set B of idempotents of $G \times A$ is a subsemigroup isomorphic with the semigroup A.

Suppose now, that S is a completely simple semigroup and that the set B of all its idempotents is a subsemigroup. If $F = \{G_\delta ; \delta \in A\}$ is the collection of all maximal (different) subgroups of S, then (as we have seen in the proof of the Theorem) if we put $a_\alpha a_\beta \in G_\gamma \Leftrightarrow$ $\Leftrightarrow \alpha\beta = \gamma$, A becomes an anticommutative semigroup. Therefore, we have $e_\alpha e_\beta = e_\gamma \Leftrightarrow \alpha\beta = \gamma$, i. e. $e_\alpha e_\beta = e_{\alpha\beta}$, whence it follows that the mapping $e_\alpha \to \alpha$ is an isomorphism of B on A. Let ε be a fixed element of A and let us put $(x_\varepsilon, \alpha) \xi = e_\alpha x_\varepsilon e_\alpha$. Then we have

$$(x_\varepsilon, \alpha) \xi = e_\alpha x_\varepsilon e_\alpha = e_\alpha x_\varepsilon e_\varepsilon e_\alpha = e_\alpha x_\varepsilon e_{\varepsilon\alpha} = x_\varepsilon \varphi_{\varepsilon, \varepsilon\alpha} \psi_{\varepsilon\alpha, \alpha} ,$$

whence, by the proof of Lemma 1.8, we find that ξ is an one-to-one mapping of $G \times A$ onto S. We also have (using Lemma 1.7)

$$[(x_\varepsilon, \alpha)(y_\varepsilon, \beta)] \xi = (x_\varepsilon y_\varepsilon, \alpha\beta) \xi = e_{\alpha\beta} x_\varepsilon y_\varepsilon e_{\alpha\beta} =$$
$$= e_\alpha e_{\varepsilon\beta} e_\varepsilon x_\varepsilon y_\varepsilon e_\varepsilon e_{\alpha\beta} e_\beta = e_\alpha e_\varepsilon x_\varepsilon y_\varepsilon e_\varepsilon e_\beta = e_\alpha x_\varepsilon y_\varepsilon e_\beta =$$
$$= e_\alpha x_\varepsilon e_{\varepsilon\alpha\beta\varepsilon} y_\varepsilon e_\beta = e_\alpha x_\varepsilon e_\varepsilon e_\alpha e_\beta e_\varepsilon y_\varepsilon e_\beta =$$
$$= e_\alpha x_\varepsilon e_\alpha \cdot e_\beta y_\varepsilon e_\beta = (x_\varepsilon, \alpha) \xi \cdot (y_\varepsilon, \beta) \xi ,$$

i. e. ξ is an isomorphism of $G \times A$ on S. This completes the proof of Corollary 1.

Clearly, Corollary 2 is a consequence of the Theorem, of Lemma 1.7 and of Corollary 1.

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[2] *A. H. Clifford,* Bands of Semigroups, Proc. Amer. Math. Soc. **5** (1954), 499—504,

[3] Е. С. Ляпин, Полугруппы, Москва, 1960.

O POTPUNO PROSTIM POLUGRUPAMA

Sadržaj

Neka je G grupa, L i R dva neprazna skupa, a λ preslikavanje skupa $R \times L$ u G. Ako stavimo

$$(x_1\,;\,l_1,\,r_1)\,(x_2\,;\,l_2,\,r_2) = (x_1\,\lambda\,(r_1,\,l_2)\,x_2\,;\,l_1,\,r_2) \quad (x_i \in G,\, l_i \in L,\, r_i \in R)\,,$$

onda skup $S = G \times L \times R$ postaje polugrupa[1], koju ćemo označiti sa $S(G\,;\,L,\,R\,;\,\lambda)$. Za polugrupu se kaže da je potpuno prosta bez nule ako je izomorfna sa nekom polugrupom oblika $S(G\,;\,L,\,R\,;\,\lambda)$; u daljem izlaganju izostavljaćemo izraz »bez nule«.

Poznato je više karakterističnih osobina potpuno prostih polugrupa (videti na primer [3], gl. V), a osnovni zadatak ovog rada je da se ukaže na još jednu karakteristiku pomenute klase polugrupa. Upravo, pokazano je da je polugrupa S potpuno prosta ako i samo ako postoji neka familija podgrupa $F = \{G_\alpha\,;\,\alpha \in A\}$, takva da su zadovoljeni sledeći uslovi: (*i*) $S = \cup\, G_\alpha$, (*ii*) $G_\alpha\, G_\beta\, G_\alpha \subseteq G_\alpha$, za svaki par $G_\alpha, G_\beta \in F$. U tom su slučaju međusobno disjunktne i izomorfne sve grupe koje pripadaju datoj familiji; osim toga imamo i $G_\alpha\, G_\beta \in F$, ako je $G_\alpha, G_\beta \in F$.

Ako je G grupa i A antikomutativna polugrupa, t. j. tačan je identitet $\alpha\beta\alpha = \alpha$ u polugrupi A, onda je direktni proizvod $G \times A$ potpuno prosta polugrupa. Pri tome je u skupu $G \times A$ operacija određena sa $(x, \alpha)\,(y, \beta) = (xy, \alpha\beta)$. Potpuno prosta polugrupa S je tog oblika, t. j. izomorfna sa direktnim proizvodom neke grupe G i antikomutativne polugrupe A, ako i samo ako je potpolugrupa polugrupe S skup svih idempotentnih elemenata te polugrupe. U tom slučaju je polugrupa A izomorfna sa polugrupom idempotentnih elemenata.

Na kraju, kao posledica prethodnih rezultata, dobija se i sledeći. Ako je polugrupa S unija neke familije podgrupa od kojih je svaka i levi ideal,[2] onda je S izomorfna nekoj polugrupi oblika $G \times A$, gde je G grupa A neprazan skup, a operacija određena sa $(x, \alpha)\,(y, \beta) = (xy, \beta)$.

[1] Polugrupom nazivamo algebarsku strukturu sa jednom binarnom asocijativnom operacijom.