

ON SEMIGROUPS S IN WHICH EACH PROPER SUBSET Sx IS A GROUP

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A semigroup S is called F_{ep} -semigroup if there is a proper subset Sx of S (i. e. $Sx \neq S$) and if every such subset is a subgroup. Clearly, every F_e -semigroup [3] (and, therefore, every F -semigroup [2]) is an F_{ep} -semigroup too. The purpose of this paper is to describe the structure of F_{ep} -semigroups.

1. Let S be an F_{ep} -semigroup and let us put:

$$P = \{x; Sx = S\}, \quad Q = \{x; x \leq Sx \neq S\}, \quad B = \{x; x \notin Sx \neq S\}.$$

Then we have: $S = Q \cup B \cup P$ and $Q \cap B = Q \cap P = B \cap P = 0$. The set Q is non-empty; if Q is not a group then it is an F_{ep} -semigroup; at least one of the sets B, P is non-empty, when Q is a group.

1.1. There is a group G and a set A such that Q is isomorphic with the semigroup $G \times A$ defined by:

$$(x, a)(y, \beta) = (xy, \beta) \quad (x, y \in G; a, \beta \in A). \quad (1)$$

Therefore, we may suppose that $Q = G \times A$. If $G_a = \{(x, a); x \in G\}$, then $\{G_a; a \in A\}$ is a collection of all the left ideals of S which are subgroups; and these groups are isomorphic with G .

1.2. Let us suppose that the set B is non-empty. Then, there is a mapping $(\varphi(b), \xi_b)$ of B in $Q (= G \times A)$ such that the following statements are satisfied:

$$(x, a)b = (x\varphi(b), \xi_b); \quad (2)$$

$$b(x, a) = (\varphi(b)x, a); \quad (x \in G; a \in A; b, c \in B) \quad (3)$$

$$bc = (\varphi(b)\varphi(c), \xi_c). \quad (4)$$

1.3. If the set P is non-empty then it is a left simple subsemigroup (i. e. $Pp = P$, for each $p \in P$) and moreover, $xp \in P \Leftrightarrow x \in P$. Then there is a homomorphism ψ of P in G and a mapping $\tau: (a, p) \rightarrow ap$ of $A \times P$ in A such that

$$a(pq) = (ap)q; \quad (5)$$

$$Ap = A; \quad (6)$$

$$(x, a)p = (x\psi(p), ap); \quad (x \in G; a \in A; b \in B; p, q \in P) \quad (7)$$

$$p(x, a) = (\psi(p)x, a); \quad (8)$$

$$pb = (\psi(p)\varphi(b), \xi_b). \quad (9)$$

1.4. Let $B \neq 0, P \neq 0$ and let us put

$$a \leq p[b] \Leftrightarrow ap = b. \quad (10)$$

Then we have

For each $p \in P$ and $b \in B$, $p[b]$ is a non-empty subset of B ; (11)

$$p[q[b]] = pq[b] \quad (\text{where } p[q[b]] = \bigcup_{c \in p[b]} p[c]); \quad (12)$$

$$b \neq c \Rightarrow p[b] \cap p[c] = 0; \quad (13)$$

$$a \leq p[b] \Rightarrow \varphi(b) = \varphi(a)\psi(p) \ \& \ \xi_b = \xi_a p. \quad (14)$$

If $ap \notin B$, then we have

$$ap = (\varphi(a)\psi(p), \xi_a p). \quad (15)$$

If there is an idempotent in P (or if the set B is finite) then in each subset $p[b]$ there is only one element, and $\{p[b]; p \in P\}$ is a collection of permutations of the set B . Therefore (in that case) the equation $xp = b$ is uniquely solvable in B , for every $p \in P, b \in B$.

2. Let G be a group, P a left simple semigroup and $A (\neq 0)$, B two sets such that $G \times A, B$ and P are mutually disjoint. Some of the sets P, B may be empty, but at least one of them is non-empty if A contains only one element. Let $(\varphi(b), \xi_b)$ be a mapping of B into $G \times A$, ψ a homomorphism of P in G and $\tau: (a, p) \rightarrow ap$ a mapping of $A \times P$ into A , such that the statements (5) and (6) are satisfied. If the sets B and P are non-empty, then let $\Gamma = \{p[b]; p \in P, b \in B\}$ be a collection of subsets of B such that the statements (11)—(14) are satisfied.

2.1. Let $S = S(G, A, B, P; \varphi, \xi, \psi, \tau; \Gamma) = G \times A \cup B \cup P$, and let the product of two elements of S be defined by (2)—(4), (7)—(10) and by (15) if $a \in p[B]$; if $p, q \in P$ then $pq = s$ in $P (\Leftrightarrow) pq = s$ in S . Then $S(G, A, B, P; \varphi, \xi, \psi, \tau; \Gamma)$ is an F_{ep} -semigroup.

2.2. For an arbitrary group G , left simple semigroup P and sets $A (\neq 0)$, B there is an $S(G, A, B, P; \varphi, \xi, \psi, \tau; \Gamma)$ -semigroup. Namely it is sufficient to put $\varphi(b) = u, \xi_b = \gamma, \psi(p) = e, ap = a$, for every $b \in B, p \in P, a \in A$, where $u \in G$ and $\gamma \in A$ are fixed elements, and e is the identity of the group G .

For an arbitrary group G , a left simple semigroup without idempotents P , and a non-empty set A , there is an $S(G, A, B, P; \varphi, \xi, \psi, \tau; \Gamma)$ -semigroup such that (for each $p \in P$ and $b \in B$) $p[b]$ contains only one element and $p[B] = \bigcup_{b \in B} p[b]$ is a proper subset of B .

I do not know whether there are semigroups $S(G, A, B, P; \varphi, \xi, \psi, \tau; \Gamma)$ such that some of the subsets $p[b]$ contain more than one element.

3. From 1. and 2. it follows:

Theorem. Every F_{ep} -semigroup is isomorphic with some $S(G, A, B, P; \varphi, \xi, \psi, \tau; \Gamma)$ -semigroup.

4. A semigroup S is called an F_{em} -semigroup if there is a proper minimal left ideal and if all such ideals are subgroups. Clearly, every F_{ep} -semigroup is an F_{em} -semigroup too.

4.1. Let S be an F_{em} -semigroup such that $SS = Q$, where $Q (= G \times A)$ is the class sum of all the minimal left ideals; and let the set $B = S \setminus Q$ be non-empty. Then there are mappings $\varphi(b)$ of B in G and $\xi(a, b)$ of $A \times B$ in A such that the following statements are valid:

$$b(x, a) = (\varphi(b)x, a); \quad (16)$$

$$(x, a)b = (x\varphi(b), \xi(a, b)); \quad (17)$$

$$ab = (\varphi(a)\varphi(b), \xi(\xi(a, a), b)); (x \in G, b \in B, a, \beta \in A) \quad (18)$$

$$\xi(\xi(a, a), b) = \xi(\xi(\beta, a), b). \quad (19)$$

Therefore $\xi(\xi(a, a), b)$ is a mapping of $B \times B$ into A .

4.2. Let G be a group and $A (\neq 0)$, B two sets. Let φ be a mapping of B into G and $\xi(a, b)$ of $A \times B$ into B , such that (19) holds. If the product of two elements of $S = G \times A \cup B$ is defined by (1), (3), (17) and (18), then S becomes an F_{em} -semigroup.

4.3. Let G be a group and u a fixed element of G , and let $A = \{a, \beta, \gamma\}, B = \{b\}$. If we put $\xi(a, b) = \xi(\beta, b) = a, \xi(\gamma, b) = \beta, \varphi(b) = u$, then (19) is satisfied, and therefore $S = G \times A \cup \{b\}$ is an F_{em} -semigroup. But S is not an F_{ep} -semigroup because $Sb = G \times A \neq S$ is not a subgroup of S .

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**O POLUGRUPAMA S U KOJIMA JE SVAKI PRAVI PODSKUP
 OBLIKA Sx PODGRUPA**

Sadržaj

Nazovimo F_{ep} -polugrupom svaku polugrupu S sa osobinom da su svi pravi podskupovi oblika Sx grupe.

Ako je S F_{ep} -polugrupa i ako stavimo $Q = \{x; x \leq Sx \neq S\}$, $B = \{x; x \leq Sx = S\}$, $P = \{x; Sx = S\}$, dobijamo $S = Q \cup B \cup P$, pri čemu su skupovi Q , B i P međusobno disjunktne. Daćemo sada opis strukture F_{ep} -polugrupa.

1°. Q je levi ideal i on je (kao polugrupa) izomorfan sa nekom polugrupom oblika $G \times A$, gde je G grupa, A neki skup i operacija određena sa (1). Zato možemo smatrati da je $Q = G \times A$.

2°. Postoji preslikavanje $b \rightarrow (\varphi(b), \xi_b)$ skupa B u Q koje zadovoljava jednačine (2)—(4).

3°. P je potpolugrupa polugrupe S . Pri tome imamo $Pp = P$, $Qp = Q$ i $B \subseteq Bp \subseteq B \cup Q$ za svako $p \in P$. Postoji F_{ep} -polugrupa gde je $B \subset Bp$ i takva gde je $B = Bp \subset B \cup Q$.

4°. Postoji homomorfizam ψ polugrupe P u grupi G i preslikavanje $(a, p) \rightarrow ap$ skupa $A \times P$ u A , koji zadovoljavaju jednačine (5)—(9). Ako za $a \in B$, $p \in P$ imamo $ap \in Q$ onda je tačna i jednačina (15).

5°. Prema 4°, za svaki par $p \in P$, $b \in B$ postoji bar jedno rešenje jednačine $xp = b$ koje se nalazi u skupu B . Ako je $p[b]$ skup svih takvih rešenja onda familija $\Gamma = \{p[b]; p \in P, b \in B\}$ tih podskupova skupa B zadovoljava uslove (12)—(14).

Ako postoji idempotentan element u polugrupi P , ili ako je konačan skup B , onda $p[b]$ sadrži samo jedan element, t. j. jednačina $xp = b$ je jednoznačno rešiva. Pri tome imamo $Bp = B$ i $p[B] = \bigcup_{b \in B} p[b] = B$. Postoji F_{ep} -polugrupa u kojoj je (za svako

$p \in P$) $p[B]$ pravi podskup skupa B , t. j. bar jedan element skupa B nije rešenje jednačine oblika $xp = b$. Nije poznat primer F_{ep} -polugrupe u kojoj neka takva jednačina ima više od jednog rešenja.

Jasno je da neki od skupova Q, B i P može biti prazan. Pri tome imamo: (i) $G \times A = 0 \Rightarrow B = 0, P = S \neq 0$, a u ovom slučaju je $Sx = S$, za svako $x \in S$; (ii) $B = 0$ ili $P = 0 \Rightarrow \Gamma = 0$.

6°. Neka je G grupa, P polugrupa sa osobinom $Pp = P$ za svako $p \in P$ i neka su B i A dva skupa; pri tome pretpostavljamo da su međusobno disjunktne skupovi $G \times A, P, B$ i dopuštamo da je neki od njih prazan, ali treba da je zadovoljen gore pomenuti uslov (i). Neka je $b \rightarrow (\varphi(b), \xi_b)$ preslikavanje skupa B u $G \times A$, ψ homomorfizam polugrupe P u grupi G i $(a, p) \rightarrow ap$ preslikavanje skupa $A \times P$ u A i neka su pri tome zadovoljene jednačine (5)—(6). Neka je $\Gamma = \{p[b]; p \in P, b \in B\}$ familija podskupova skupa B koja zadovoljava uslove (12)—(14).

Skup $S = G \times A \cup B \cup P$ postaje F_{ep} -polugrupa ako se operacija odredi sa: (i) $p_1 p_2 = p$ u $P \Leftrightarrow p_1 p_2 = p$ u S ; (ii) jednačinama (1)—(4), (7)—(9) i (10) ili (15) u zavisnosti od toga da li je $a \in p[B]$ ili $a \notin p[B]$.

Prema tome, sa 1°—5° je potpuno opisana struktura klase F_{ep} -polugrupa.

F_{ep} -polugrupe čine potklasu klase polugrupa u kojima su svi minimalni levi ideali i podgrupe. U radu je dat primer polugrupe koja nije F_{ep} -polugrupa iako su svi minimalni levi ideali grupe.