ON TOPOLOGICAL N-GROUPS

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In this note it is shown that each topological n-group can be embedded into a topological group.

At first, some preliminary definitions and results will be stated.

Let Q be a non-empty set and:

$$\begin{aligned} (\ldots) &: (x_0, \ldots, x_n) \to (x_0 \cdots x_n) \\ [..\diagdown] &: (x_0, \ldots, x_n) \to [x_n \cdots x_1 \diagdown x_0] \\ [\diagup..] &: (x_0, \ldots, x_n) \to [x_0 \diagup x_n \cdots x_2] \end{aligned} \tag{1}$$

be three $n+1$-ary operations on Q. Q is said to be an n-group if the following identities are satisfied.

$$((x_0 \cdots x_n) \cdots x_{2n}) = (x_0 (x_1 \cdots x_{n+1}) \cdots x_{2n}) = \ldots = (x_0 \cdots (x_n \cdots x_{2n})) \tag{2}$$

$$(x_1 \cdots x_n [x_n \cdots x_1 \diagdown x]) = x = ([x \diagup x_n \cdots x_1] x_1 \cdots x_n). \tag{3}$$

An n-group Q is said to be a topological n-group if Q is a topological space such that the operations $(\ldots)$, $[..\diagdown]$ and $[\diagup..]$ are continous (in the all variables together).

The following result is known as Post's Coset Theorem[1]). If Q is an n-group then there is a unique (within to a canonical isomorphism) group G such that:

$$G = Q \cup Q^2 \cup \ldots \cup Q^n \tag{4}$$

$$(\forall a_0, \ldots, a_n \in Q)(a_0 \cdots a_n) = a_0 \cdots a_n \tag{5}$$

$$1 \leqslant i \leqslant j \leqslant n,\ a_\nu,\ b_\lambda \in Q \Rightarrow \{a_1 \ldots a_i = b_1 \ldots b_j \Leftrightarrow \tag{6}$$
$$\Leftrightarrow [i = j \ \&\ (\exists c_0, \cdots, c_{n-i} \in Q)(c_0 \cdots c_{n-i} a_i \cdots a_i = (c_0 \cdots c_{n-i} b_1 \cdots b_i)]\}.$$

The group G is said to be the free covering of the given n-group Q

Now we shall state and prove the following

Theorem. Let Q be a toopological n-group, and G the free covering of Q. Define a collection $\mathcal{B}$ of subsets of G by:

$$\mathcal{B} = \{A_1 \cdots A_k \mid 1 \leqslant k \leqslant n,\ A_1, \ldots, A_k \text{ are open in } Q\} \tag{7}$$

Then we have:

(i) $\mathcal{B}$ is a base of a topology $\mathcal{G}$ on G.

(ii) The given topology on Q is induced by $\mathcal{G}$ on Q, and Q is an open and closed subset of G.

(iii) G is a topological group.

(iv) If Q is a compact (Hausdorff) n-group then G is a compact (Hausdorff) group.

First, we prove three lemmas.

1. If $A_0, \ldots, A_n \subseteq Q$, and A_n is an open subset of Q, then $(A_0 \cdots A_n) = A$ is an open subset of Q too.

Proof. If $a_1, \ldots, a_n$ are arbitrary elements of Q then the mappings: $f(x) = (a_1 \cdots a_n x)$ and $f^{-1}(x) = [a_n \cdots a_1 \diagdown x]$ are continous, and therefore $f(x)$ is a homeomorphism.

2. Let $a_1, \ldots, a_i$, $b_1, \ldots, b_i \in Q$, $1 \leqslant i \leqslant n$ and $B_1, \ldots, B_i$ be (open) neighborhoods of $b_1, \ldots, b_i$. If $a_1 \cdots a_i = b_1 \cdots b_i$ (in G) then there exist neighborhoods $A_1, \ldots, A_i$ of $a_1, \ldots, a_i$ such that $A_1 \cdots A_i \subseteq B_1 \cdots B_i$.

Proof. By (6) there exist $c_0, \ldots, c_{n-i} \in Q$ such that $c = (c_0 \cdots c_{n-i} a_1 \cdots a_i) = (c_0 \cdots c_{n-i} b_1 \cdots b_i)$, and (by 1) $U = (c_0 \ldots c_{n-i} B_1 \ldots B_i)$ is a neighborhood of c. Therefore there exist neighborhoods $C_0, \ldots, C_{n-i}$ of $c_0, \ldots, c_{n-i}$, and $A_1, \ldots, A_i$ of $a_1, \ldots, a_i$ such that

$$(C_0 \cdots C_{n-i} A_1 \cdots A_i) \subseteq U = (c_0 \cdots c_{n-i} B_1 \cdots B_i)$$

i. e.

$$c_0 \cdots c_{n-i} A_1 \cdots A_i \subseteq C_0 \cdots C_{n-i} A_1 \cdots A_i \subseteq c_0 \cdots c_{n-i} B_1 \cdots B_i,$$

and thus we have $A_1 \ldots A_i \subseteq B_1 \ldots B_i$.

3. If $a_1, \ldots, a_i$, $b_1, \ldots, b_{n-i} \in Q$, and $s = a_1 \cdots a_i$ (in G), then

$$s^{-1} = [b/b_1 a_i \cdots a_1 b_{n-i} \cdots b_2] b_2 \cdots b_{n-i}. \tag{8}$$

Proof. By (3) and (5) we have $[x/x_n \cdots x_1] = x x_n^{-1} \cdots x_1^{-1}$, and this implies (8).

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1) [1] p. 37 , or [5] p. 218

Proof of Theorem. (i) By (4) and (7), $G = \underset{B \in \mathcal{B}}{U} B$. Assume that $g \in A_1 \ldots A_i \cap B_1 \ldots B_i$, where $1 \leqslant i \leqslant n$, and A_ν, B_ν are open in Q. Then there exist $a_\nu \in A_\nu$, $b_\nu \in B_\nu$, such that $g = a_1 \cdots a_i = b_1 \cdots b_i$. By **2**, there exist neighborhoods $A_1', \ldots, A_i'$ of $a_1, \ldots, a_i$ such that $A_1' \cdots A_i' \subseteq B_1 \cdots \cdots B_i$, and thus $g \in A_1'' \cdots A_i'' \subseteq A_1 \cdots A_i \cap B_1 \ldots B_i$, where $A_\nu'' = A_\nu \cap A_\nu'$. This proves that $\mathcal{B}$ is a base of a topology $\mathcal{T}$ on G.

(ii) By (6) and (7), $Q, Q^2, \ldots,$ and Q^n are disjoint open subsets of G, and therefore they are closed too.

If $A \subseteq Q$ and if A is an open set in the given topolology on Q, then $A \in \mathcal{B}$,i. e. A is open in G too. Convercely, if A is an open subset of G and $A \subseteq Q$, then $A = \underset{i \in I}{U} B_i$, where $B_i \in \mathcal{B}$; by (6), for each $i \in I$, there exist open sets $A_1, \ldots, A_{k_i}$ $(1 \leqslant k_i \leqslant n)$ such that $B_i = A_1 \cdots A_{k_i}$, and this, implies that $k_i = 1$, i. e. $B_i = A_1$ is an open subset of Q, and therefore A is an open subset of Q too. This proves fhat the given topology on Q is induced by $\mathcal{T}$.

(iii) Let $s = a_1 \cdots a_i$, $t = b_1 \cdots b_j$ $(i, j \leqslant n,\ a_\nu, b_\lambda \in Q)$ be two elements of G, and $g = st$. Let $C \in \mathcal{B}$ and $g \in C$. Then, if $C = C_1 \cdots C_k$ $(k \leqslant n)$, where $C_1, \ldots, C_k$ are open sets in Q, there exist $c_1, \ldots, c_k \in Q$ such that $c_\nu \in C_\nu$, and $g = c_1 \cdots c_k$. Thus:

$$a_1 \cdots a_i\, b_1 \cdots b_j = c_1 \cdots c_k.$$

If $i + j \leqslant n$, then $i + j = k$, and by **1** there exist neighborhoods $A_1, \ldots, A_i$ of $a_1, \ldots, a_i$ and $B_1, \ldots, B_j$ of $b_1, \ldots, b_j$ such that

$$A_1 \cdots A_i\ B_1 \cdots B_j \subseteq C_1 \cdots C_k = C, \tag{9}$$

where $A = A_1 \cdots A_i$ is a neighborhood of s and $B = B_1 \cdots B_j$ is a neighborhood of t.

If $i + j > n$ and if we put $a = (a_1 \cdots a_i\, b_1 \cdots b_{n-i+1})$, then we have $a\, b_{n-i+2} \cdots b_j = c_1 \cdots c_k$ and $k = i + j - n$; by **1** there exist neighborhoods A' of a and $B_{n-i+2}, \ldots, B_j$ of $b_{n-i+2}, \ldots, b_j$ such that $A'\, B_{n-i+2} \ldots B_j \subseteq C$. From the equation $a = (a_1 \cdots a_i\ b_1 \cdots b_{n-i+1})$ follows that there exist neighorhoods $A_1, \ldots, A_i$ of $a_1, \ldots, a_i$ and $B_1, \ldots, B_{n-i+1}$ of $b_1, \cdots, b_{n-i+1}$ such that $(A_1 \ldots A_i B_1 \ldots B_{n-i+1}) \subseteq A'$, and this implies that $AB \subseteq C$, where $A = A_1 \cdots A_i$, $B_1 \cdots B_j = B$. This completes the proof that the operation "$\cdot$" of the group G is continous.

Assume now that $s = a_1 \cdots a_i \in G$, where $1 \leqslant i \leqslant n$, $a_r \in Q$. If $b_1, \ldots b_{n-i}$ are arbitrary elements of Q, then by **3** we have $s^{-1} = bb_2 \cdots b_{n-1}$, where $b = [b_1/b_1\, a_i \cdots a_1\, b_{n-i} \cdots b_2]$. If $C = B'\, C_2 \cdots C_{n-i}$ is a neighborhood of s^{-1}, then $s^{-1} = b'\, c_2 \cdots c_{n-i} = bb_2 \cdots b_{n-i}$, where $b' \in B'$, $c_\nu \in C_\nu'$. By **2** there exist neighborhoods B, $B_2, \ldots, B_{n-i}$ of $b, b_2, \ldots, b_{n-i}$ such that $BB_2 \ldots B_{n-i} \subseteq C$. From $b \in B$ and $b = [b_1/b_1\, a_i \cdots a_1\, b_{n-i} \cdots b_2]$ follows that there exist neighborhoods B_1', B_2'' of b_1, $A_1, \ldots, A_i$ of $a_1, \ldots, a_i$ and $B_2', \ldots, B'_{n-i}$ of $b_2', \ldots, b'_{n-i}$, such that $[B_1'/B_1''\, A_1 \cdots \cdots B_2'] \subseteq B$. Then we have:

$$(A_1 \cdots A_i)^{-1} = [b_1/b_1\, A_i \cdots A_1\, b_{n-i} \cdots b_2]\, b_2 \ldots b_{n-i} \subseteq$$
$$\subseteq [B_1'/B_1''\, A_i \cdots A_1\, B_{n-i} \cdots B_2]\, B_2 \cdots B_{n-i} = C.$$

Thus we have proved that the mapping $s \to s^{-1}$ is continous, and this completes the proof of the statement (iii).

(iv) If the given space Q is compact then each cartezian product $Q \times Q \times \cdots \times Q$ is a compact space, and hence Q^k is a compact subset of G, because the mapping $(x_1, \ldots, x_k) \to x_1 \cdots x_k$ is continous. Then G is a compact space for it is a union of n compact subsets $Q, Q^2, \ldots, Q^n$.

Assume now Q to be a Hausdorff space, and let $s = a_1 \cdots a_i$, $t = b_1 \cdots b_j$ $(a_\nu, b_\lambda \in Q,\ 1 \leqslant i \leqslant j \leqslant n)$ be two different elements of G. If $i \neq j$, then Q^i is a neighborhood of s, and Q^j is a neighborhood of t, where $Q^i \cap Q^j = \emptyset$. If $i = j$, then for arbitrary $c_0, \ldots, c_{n-i} \in Q$ we have:

$$(c_0 \cdots c_{n-i}\ a_1 \cdots a_i) = a \neq b = (c_0 \cdots c_{n-i}\ b_1 \cdots b_i),$$

and fherefore there are a neighborhood A of a and a neighborhood B of b such that $A \cap B = \emptyset$. Also, there exist neighborhoods $C_0, \ldots, C_{n-i}$ of

$c_0, \ldots, c_{n-i}$, $A_1, \ldots, A_i$ of $a_1, \ldots, a_i$ and $B_1, \ldots, B_i$ of $b_1, \ldots, b_i$ such that

$$(C_0 \cdots C_{n-i} A_1 A_i) \subsetneq A, \ (C_0 \cdots C_{n-i} B_1 \cdots B_i) \subseteq B,$$

whence follows

$$c_0 \cdots c_{n-i} \ A_1 \cdots A_i \cap c_0 \cdots c_{n-i} \ B_1 \cdots B_i = \emptyset,$$

i. e. $A \cap B = \emptyset$, where $A = A_1 \cdots A_i$ is a neighborhood of s, and $B = B_1 \cdots B_i$ is a neighborhood of t. This completes the proof that G is a Hausdorff group.

Some remarks

a) It is well known ([5], p. 21) that the notions of T_0, T_1 and T_2 spaces are equivalent in the class of topological groups. Is the same statement true in the class of topological n-groups?

b) An algebra $Q(\ldots)$ with an $n+1$ —ary operation $(\ldots)$ is said to be an n-semigroup if the identities (2) are satisfied. Then ([2], p. 23) there is a semigroup G (the free covering semigroup of Q) such that (4) and (5) are satisfied. If in addition the operation $(\ldots)$ is continous (in the all variables together) then Q is said to be a topological n-semigroup Are the statements of Theorem true in the class of topological n-semigroups?

It is known ([3]) that every topological universal algebra $A(\Omega)$ may be (in a corresponding way) embedded into a topological semigroup S, but even in the case when A is a topological n-semigroups, S is not the free covering of A.

c) A group G is said to be semitopological if G is a topological space in which each translation (left or right) is continous. This notion for n- groups. can be generalized in an obvious way. Are the statements of Theorem true for semitopological n-groups?

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ЗА ТОПОЛОШКИТЕ n-ГРУПИ

Резиме

Во работава се докажува следнава

Теорема. Нека Q е тополошка n-група, а G групата што е слободна покривка на Q. Ако $\mathcal{B}$ е фамилијата подмножества на G определена со:

$$\mathcal{B} = \{A_1 \cdots A_k \mid A_1, \ldots, A_k \text{ се отворени во } Q\}$$

тогаш имаме:

(i) $\mathcal{B}$ е база на топологија $\mathcal{T}$ над G, при што дадената топологија над Q е индуцирана од $\mathcal{T}$. Q е и отворено и затворено подмножество во G.

(ii) G е тополошка група во однос на топологијата $\mathcal{T}$.

(iii) Ако тополошката n-група Q е компактна (Хаусдорфова), тогаш и тополошката група G е компактна (Хаусдорфова).

(Притоа, за алгебрата Q со три $n+1$— арни операции $(\ldots)$ $[\ldots\backslash]$, $[/\ldots]$ се вели дека е n-група, ако се исполнети идентитетите (2) и (3); групата G е слободна покривка на n-групата Q ако се исполнети условите (4), (5) и (6); n-групата Q се вика тополошка ако Q е тополошки простор, таков што операциите на n-групата се непрекинати.)