ON n-GROUPOIDS

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An algebra $Q(f)$ with an n-ary operation is said to be an n-s u b-g r u p o i d of a grupoid $G(\bullet)$ if $Q \subseteq G$ and f is the restriction of $\bullet^{n-1}$ on Q. And, an algebra $A(F)$ is said to be an F-g r o u p o i d if there is a groupoid $G(\bullet)$ such that $A(f)$ is an n-subgroupoid of $G(\bullet)$ for every n-ary operator $f \in F$. It is shown in § 1 that every n-groupoid is an n-subgroupoid of a groupoid. The classes of n-subgroupoids of each of the classes of cancellative groupoids and commutative groupoids are described in §§ 2,3. It is shown in § 4 that the class of F-groupoids is a variety iff there is an n-ary operator $f \in F$ such that, for every m-ary operator $g \in F$, $n-1$ is a divisor of $m-1$.

1. Universal covering groupoids. An algebra $Q(f)$ with an n-ary operation is said to be an n-g r o u p o i d, and it is an n-subgroupoid of a groupoid $G(*)$ if $Q \subseteq G$ and

$$fa_1 \dots a_n = *^{n-1}\, a_1 \dots a_n,$$

for all $a_1, \dots, a_n \in Q$. The following result can be obtained as a corrolary from the main results of the papers [4] and [6], but we shall give here a direct proof.

1.1. *Every n-groupoid is an n-subgroupoid of a groupoid.*

Proof. Let $Q(f)$ be an n-groupoid and $W(\mathrm{o})$ be the groupoid which is freely generated by the set Q. Thus, W is the minimal set of finite sequences on $Q \cup \{\mathrm{o}\}$ (where $\mathrm{o} \notin Q$) satisfyng the following statements:

(i) $Q \subseteq W$; (ii) $u, v \in W \Rightarrow \mathrm{o}\, uv \in W$.

Denote by U the set of elements of W in which do not occur subsequences of the following form:

$$\mathrm{o}^{n-1} a_1 \dots a_n \quad (a_1, \dots, a_n \in Q).$$

Define a binary operation $\bullet$ on U by:

$$u, v, \mathrm{o}\, uv \in U \Rightarrow \bullet\, uv = \mathrm{o}\, uv$$

and

$$b = fa_1 \dots a_{n-1} a_n \text{ in } Q(f) \Rightarrow \bullet\ \mathrm{o}^{n-2}\, a_1 \dots a_{n-1}\, a_n = b.$$

Clearly, $Q(f)$ is an n-subgroupoid of the groupoid $U(\bullet)$. □

The groupoid $U(\bullet)$ is said to be the u n i v e r s a l c o v e r i n g g r o u p o i d of the n-groupoid $Q(f)$. It is easy to see that the following propositions hold.

1.2. *If $u, v, u', v' \in U$ are such that $\bullet uv = \bullet u' v'$* and ($u \neq u'$ or $v \neq v'$), *then $v, v' \in Q$ and:*

$$u = \mathrm{o}^{n-2}\, a_1 \dots a_{n-1},\ u' = \mathrm{o}^{n-2}\, a'_1 \dots a'_{n-1},$$

$$f\, a_1 \dots a_{n-1}\, v = f\, a'_1 \dots a'_{n-1}\, v'.$$

for some $a_i, a_i' \in Q$. □

1.3. *If $u_1, \dots, u_n \in U$, then: $\bullet^{n-1} u_1 \dots u_n \in Q \Leftrightarrow u_1, \dots, u_n \in Q$.* □

1.4. *Let $\Pi\,(a_1, \dots, a_m)$ be a continued product in $U(\bullet)$, and $a_1, \dots, a_m \in Q$. Then $\Pi\,(a_1, \dots, a_m) \in Q$ iff there is a continued product $\Pi'\,(a_1, \dots, a_m)$ in $Q(f)$ such that Π is obtained form Π' replacing each occuring of an operator sign f by $\bullet^{n-1}$. Then we also have:*

$$\Pi'\,(a_1, \dots, a_m) = \Pi\,(a_1, \dots, a_m). \ \square$$

2. n-subgroupoids of groupids with cancellation. First we state some definitions. Let $Q(f)$ be an n-groupoid and $i \in \{1, \dots, n\}$. We say that $Q(f)$ is i-c a n c e l l a t i v e if the following quasiidentity is satisfied:

$$fx_1 \dots x_{i-1}\, yx_{i+1} \dots x_n = fx_1 \dots x_{i-1}\, zx_{i+1} \dots x_n \Rightarrow y = z.$$

And, $Q(f)$ is c a n c e l l a t i v e if it is i-cancellative for each i. Instead of n-c a n c e l l a t i v e (1-c a n c e l l a t i v e) we shall say l e f t c a n-c e l l a t i v e (r i g h t c a n c e l l a t i v e).

From **1.2.** follows that the universal covering groupoid of a left cancellative n-groupoid is left cancellative groupoid, and this implies the following proposition:

2.1. ***The class of left cancellative n-groupoids and the class of n-subgroupoids of left cancellative groupoids are equal.*** □

It is easy to see that if $Q(f)$ is an n-subgroupoid of a cancellative (right cancellative) groupoid, then $Q(f)$ is cancellative (right cancellative) and the following quasiidentity is satisfied in $Q(f)$:

$$fx_1 \dots x_i z_1 \dots z_{n-i} = fy_1 \dots y_i z_1 \dots z_{n-i} \Rightarrow \qquad (2.1.)$$
$$fx_1 \dots x_i u_1 \dots u_{n-i} = fy_1 \dots y_i u_1 \dots u_{n-i},$$

for each $i \in \{1, \dots n\}$.

Conversely, assume that $Q(f)$ is a right cancellative n-groupoid in which all the quasiidentities (2.1) are satisfied. The universal covering $U(\bullet)$ of $Q(f)$ can be not right cancellative. We are asking for a congruence α such that $Q(f)$ can be embedded as an n-subgroupoid in $U/_\alpha(\bullet)$ and $U/_\alpha(\bullet)$ should be right cancellative.

First, for each $i \in \{1, \dots, n-1\}$, let Q be defined by:

$$Q_i = \{o^{i-1} a_1 \dots a_i \mid a_1, \dots, a_i \in Q\},$$

and let α_i be a relation in Q_i defined by:

$$o^{i-1} a_1 \dots a_i \;\; \alpha_i \;\; o^{i-1} b_1 \dots b_i \Leftrightarrow$$
$$(\exists\, c_{i+1}, \dots, c_n \in Q) fa_1 \dots a_i c_{i+1} \dots c_n = fb_1 \dots b_i c_{i+1} \dots c_n.$$

By (2.1), the quantifier $\exists$ may be changed by $\forall$, and this implies that: $c \in Q$, $o^{i-1} a_1 \dots a_i \;\; \alpha_i \;\; o^{i-1} b_1 \dots b_i \Rightarrow o^i a_1 \dots a_i c \;\; \alpha_{i+1} \;\; o^i b_1 \dots b_i c$. We also note that α_1 is the equality on Q $(= Q_1)$.

Denote by α the minimal relation on U which satisfy the following propositions:

$$\alpha_1 \cup \alpha_2 \cup \dots \cup \alpha_{n-1} \subseteq \alpha,$$
$$u_1 \,\alpha v_1,\; u_2 \alpha v_2,\; o\, u_1 u_2,\; o\, v_1 v_2 \in U \Rightarrow o u_1 u_2 \;\; \alpha \;\; o\, v_1 v_2.$$

It is easy to see that α is a congruence on $U(\bullet)$ and that $G(\bullet) = U/_\alpha(\bullet)$ is a right cancellative groupoid. Moreover, if $Q(f)$ is cancellative, then $G(\bullet)$ is cancellative too.

Finaly, the mapping $a \to a^\alpha$ embeds $Q(f)$ into $G(\bullet)$.

Thus we obtain the following result.

2.2. *An n-grupoid $Q(f)$ is an n-subgroupoid of a* (*right*) *cancellative groupoid iff $Q(f)$ is* (*right*) *cancellative and satisfies all the quassidentities* (2.1). □

As a corollary of **2.2** and the fact that every groupoid with cancellation is a subgroupoid of a quasigroup ([1], VII. 4) we obtain the following result.

2.3. *The class of n-subgroupoids of quasigroups and the class of n-subgroupoids of groupoids with cancellation are equal.* □

If $n \geqslant 3$, then there exist n-quasigroups which do not satisfy some of the quasiidentities (2.1) (for example, [7] p. 115), and thus we get the following result.

2.4. If $n \geqslant 3$, then there exist n-quasigroups which are not n-subgroupoids of quasigroups. □

3. Commutative *n*-groupoids. An n-groupoid $Q(f)$ is said to be (i, j)-c o m m u t a t i v e if:

$$fx_1 \dots x_i \dots x_j \dots x_n = fx_1 \dots x_j \dots x_i \dots x_n$$

is an identity equation. $Q(f)$ is called c o m m u t a t i v e if it is (i, j)-commutative for each pair (i, j): $1 \leqslant i < j \leqslant n$.

3.1. *An n-groupoid $Q(f)$ is an n-subgroupoid of a commutative groupoid iff $Q(f)$ is* (1, 2)*-commutative.*

Proof. Let $Q(f)$ be a (1, 2)-commutative n-groupoid, and let $C(\circ)$ be the freely generated commutative groupoid by the set Q. Then

$$\circ uv = \circ u' v' \Leftrightarrow (u = u',\ v = v') \text{ or } (u = v',\ v = u').$$

Denote by D the set of elements of C which can not be represented as products of the form:

$$\Pi (a_1, \ldots, a_{i-1}, \circ^{n-1} b_1 \ldots b_n, a_{i+1}, \ldots a_m)$$

with a_i, $b_j \in Q$.

Define a binary operation $\bullet$ on D by:

$$u, v, \circ uv \in D \Rightarrow \bullet uv = \circ uv,$$

and

$$u, v \in D, \ \circ uv = \circ^{n-1} a_1 \ldots a_n, \ a = fa_1 \ldots a_n \Rightarrow \bullet uv = a.$$

It is easy to see that:

(i) the operation $\bullet$ is well defined:

(ii) $D(\bullet)$ is a commutative groupoid;

(iii) $Q(f)$ s an n-subgroupoid of $D(\bullet)$.

It is clear that every n-subgroupoid of a commutative groupoid is a (1,2)-commutative n-groupoid. □

A groupoid $G(*)$ is said to be n-*commutative* if the n-groupoid $G(*^{n-1})$ is commutative.

3.2. *The class of n-subgroupoids of n-commutative groupoids and the class of commutative n-groupoids are equal.*

Proof. First, it is clear that every n-subgroupoid of an n-commutative groupoid is a commutative n-groupoid.

Let $Q(f)$ be a commutative n-groupoid and let $U(\bullet)$ be the universal covering groupoid of $Q(f)$. Define a relation α on U in the following way. If $v \to i_v$ is a permutation of $\{1, \ldots, n\}$, Π a product on $U(\bullet)$, and u_i, $t_j \in U$, then:

$$\Pi (u_1, \ldots, u_{p-1}, \bullet^{n-1} t_1 \ldots t_n, u_{p+1}, \ldots) \ \alpha$$
$$\Pi (u_1, \ldots, u_{p-1}, \bullet^{n-1} t_{i_1} \ldots t_{i_n}, u_{p+1}, \ldots).$$

It is obvious that the transitive and reflexive extension β of the relation α is a congruence on $U(\bullet)$ and that the groupoid $U/_{\beta}(\bullet) = G(\bullet)$ is n-commutative.

From **1.2** and **1.3** it follows that:

$$a \in Q,\ u \in U \Rightarrow (a \,\alpha\, u \Rightarrow a = u),$$

and this implies that:

$$a, b \in Q \Rightarrow (a \ \beta \ b \Rightarrow a = b),$$

i.e that $Q(f)$ can be embedded in $G(\bullet)$ as an n-subgroupoid. □

The statements **3.1** and **3.2** imply that every commutative n-groupoid is an n-subgroupoid of a commutative groupoid, and also an n-subgroupoid of an n-commutative groupoid. But there exist commutative n-groupoids which can not be embedded in groupoids which are both commutative and n-commutative, for commutativity and n-commutativity imply some associativity. (For example, every commutative and 3-commutative groupoid is a semigroup.)

4. F-groupoids. Here we assume that F is a nonempty set of finitary operators such that $F_0 \cup F_1 = \emptyset$, where F_n is the set of n-ary operators belonging to F. An algebra $A(F)$ is said to be an F-*groupoid* if there is a groupoid $G(*)$ such that $A(f)$ is an n-subgroupoid of $G(*)$ for every n-ary operator $f \in F$; then we also say that $A(F)$ is an F-subgroupoid of $G(*)$. An algebra $A(F)$ is said to be a *weak* F-*groupoid* if for every sequence of operators $f_1, \ldots, f_r, g_1, \ldots, g_s \in F$ such that:

$$f_i \in F_{n_i+1},\ g_j \in F_{m_j+1},\ n_1 + \ldots + n_r = n = m_1 + \ldots + m_s \tag{4.1}$$

the following identity is satisfied in $A(F)$:

$$f_1 \ldots f_r\, x_0 \ldots x_n = g_1 \ldots g_s\, x_0 \ldots x_n . \tag{4.2}$$

4.1. *Every F-groupoid is a weak F-groupoid.*

Proof. Let $A(F)$ be an F-subgroupoid of a groupoid $G(*)$, and assume that (4.1) is satisfied. If $a_0, \ldots, a_n \in A$ then we have:

$$f_1 \ldots f_r\, a_0 \ldots a_n = *^n\, a_0 \ldots a_n = g_1 \ldots g_s\, a_0 \ldots a_n,$$

i.e. (4.2) is an identity in $A(F)$. □

Let J_F be the following set of integers:

$$J_F = \{n \mid F_{n+1} \neq \emptyset\},$$

and denote by d_F the greatest common divisor of the numbers belonging to J_F.

4.2. *Every weak F-groupoid is an F-groupoid iff* $d_F \in J_F$.

Proof. Let $d = d_F \in J_F$, $f \in F_{d+1}$ and let $A(F)$ be a weak F-groupoid. By **1.1,** $A(f)$ is a $d+1$-subgroupoid of a groupoid $U(\bullet)$. If $g \in F_{m+1}$, then d is a divisor of m, and by (4.2) we have:

$$g x_0 \ldots x_m = f^{m/d}\, x_0 \ldots x_m = \bullet^m\, x_0 \ldots x_m,$$

and this implies that $A(F)$ is an F-subgroupoid of $U(\bullet)$.

Assume now that $d_F \notin J_F$. Then, if n is the least element of J_F there is an element $m \in J_F$ which is not divisible by n, and we shall assume that m is the least element of J_F with that property. Define an algebra $A(F)$ in the following way:

(i) $A = \{a, b, c\}$, $a \neq b \neq c \neq a$;

(ii) $f \in F_{m+1} \Rightarrow f a_0 \ldots a_m = a$ if $a_\nu \neq c$ for some ν and $f c^{m+1} = b$;

(iii) $g \in F_{k+1}$, $k \neq m \Rightarrow g a_0 \ldots a_k = a$.

It is easy to see that $A(F)$ is a weak F-groupoid. $A(F)$ is not an F-groupoid, for if $A(F)$ were an F-subgroupoid of a groupoid $G(*)$ and if $f \in F_{m+1}$, $g \in F_{n+1}$ then we would have:

$$b = f c^{m+1} = *^{m-n}\, g\, c^{m+1} = *^{m-n}\, a\, c^{m-n} =$$
$$= *^{m-n}\, g a^{n+1}\, c^{m-n} = *^m\, a^{n+1}\, c^{m-n} = f a^{n+1}\, c^{m-n} = a. \; □$$

If Σ is a class of groupoids we can ask for an axiom system of the class of F-subgroupoids of Σ-grupoids. We note that there are known convenient descriptions of F-sugroupoids of semigroups ([2], **5**), and F-subgroupoids of cancellative semigroups ([5], **3**).

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ЗА *n*-ГРУПОИДИТЕ

(Р е з и м е)

Во работава се покажува дека секој n-групоид е n-подгрупоид на групоид. Се дава опис на класата n-подгрупоиди на групоиди со кратење како и на класата n-подгрупоиди од комутативни групоиди. Се разгледува и поопштото прашање за сместување на произволни алгебри во групоиди и се докажува дека класата F-групоиди е многократност акко постои n-арен оператор $f \in F$ таков што $n - 1$ е делител на $m - 1$ за секој m — арен оператор $g \in F$.