n-SUBSEMIGROUPS OF PERIODIC SEMIGROUPS

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An algebra $Q[\;]$ with an $n+1$-ary operation

$$[\;]: (x_0, x_1, \ldots, x_n) \to [x_0\, x_1 \ldots x_n]$$

is said to be an n-subsemigroup of a semigroup S if $Q \subseteq S$ and

$$[a_0\, a_1 \ldots a_n] = a_0\, a_1 \ldots a_n$$

for all $a_0, \ldots a_n \in Q$.

A description of the class of n-subsemigroups of periodic semigroups is given in this note.

0. First we state some preliminaries.

0.1. $Q[\;]$ is an n-subsemigroup of a semigroup iff[1]) $Q[\;]$ is an n-semigroup, i.e. an algebra with an associative $n+1$-ary operation. Every commutative n-semigroup is an n-subsemigroup of a commutative semigroup. (These results, and convenient descriptions of some other classes of n-semigroups, can be found in [2]). Further on we shall usually say „n-semigroup Q“ instead of „n-semigroup $Q[\;]$“. A continued product on a sequence $(a_0, a_1, \ldots, a_{sn}) \in Q^{sn+1}$ will be denoted by $[a_0\, a_1 \ldots a_{sn}]$; if $s = 0$, then $[a] = a$.

0.2. A semigroup S is said to be periodic iff every cyclic subsemigroup is finite. Then, to every element $a \in S$ are associated two positive integers $r = r_a$ (the index of a), $m = m_a$ (the period of a) such that:

$$a^i = a^j \Leftrightarrow i = j \text{ or } (i \geqslant r,\ j \geqslant r,\ i \equiv j \pmod m)$$

(For example, [1], p. 19.) A periodic semigroup is said to have a finite index (period) iff the set of indices (periods) is finite. An n-semigroup Q is said to be periodic iff every cyclic subalgebra of Q is finite.

0.3. Usually, by $x, y, \ldots, x_1, \ldots$ will be denoted individual variables, and by $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ sequences of variables. If $\mathbf{x} = x_1 \ldots x_k$, then $|\mathbf{x}| = k$.

1. The main result of this note is the following:

Theorem. An n-semigroup Q is an n-subsemigroup of a periodic semigroup iff the following statement is satisfied:

$$(\forall\, \mathbf{x})\ (\exists\, r \geqslant 0,\ m > 0)\ (\forall\, \mathbf{y}, \mathbf{z})\ [\mathbf{y}\, \mathbf{x}^r\, \mathbf{z}] = [\mathbf{y}\, \mathbf{x}^{r+mn} \mathbf{z}]^{2)}. \qquad (1.1)$$

Proof. Let Q be an n-subsemigroup of a periodic semigroup S. If $a_1, \ldots, a_k \in Q$ and if r is the index and m the period of $b = a_1 \ldots a_k$, then $b^r = b^{r+mn}$, and this implies that

$$[\mathbf{y}\, b^r\, \mathbf{z}] = [\mathbf{y}\, b^{r+mn}\, \mathbf{z}]$$

for every pair $\mathbf{y}$, $\mathbf{z}$ of sequences of elements of Q such that $rk + |\mathbf{y}| + |\mathbf{z}| \equiv 1 \pmod n$. Thus, Q satisfies the condition (1.1).

Assume now that the n-semigroup Q satisfies the statement (1.1). Let $Q^\wedge$ be the universal covering semigroup of Q ([3] p. 25). We note that Q is a generating n-subsemigroup of $Q^\wedge$ and that the following statement is satisfied:

$$a_0, \ldots, a_s, b_0, \ldots, b_t \in Q,\ a_0 \ldots a_s = b_0 \ldots b_t \Rightarrow s \equiv t \pmod n. \qquad (1.2)$$

By (1.1), for every sequence $\mathbf{a} = (a_1, \ldots, a_k) \in Q^k$, there exist $r = r_a \geqslant 0$, $m = m_a > 0$ such that

$$[\mathbf{y}\, \mathbf{a}^r\, \mathbf{z}] = [\mathbf{y}\, \mathbf{a}^{r+mn}\, \mathbf{z}]$$

for every pair $\mathbf{y}$, $\mathbf{z}$ of sequences of elements of Q such that $rk + |\mathbf{y}| + |\mathbf{z}| \equiv 1 \pmod n$. (We can assume that r_a, m_a are the least numbers with the mentioned property).

[1]) „iff“ means „if and only if“.

[2]) „$[\mathbf{x}] = [\mathbf{y}]$“ is an abbrevation for „$1 = |\mathbf{x}| \equiv |\mathbf{y}| \pmod n \Rightarrow [\mathbf{x}] = [\mathbf{y}]$“.

Let τ_0 be a relation defined in $Q^\wedge$ in the following way. If $a_1, \ldots, a_k, c_1, \ldots, c_q \in Q$ and $1 \leqslant j \leqslant q+1$, then

$$c_1 \ldots c_{j-1} (a_1 \ldots a_k)^r c_j \ldots c_q \tau_0 c_1 \ldots c_{j-1} (a_1 \ldots a_k)^{r+mn} \quad c_j \ldots c_q,$$

where r and m are the numbers which are associated to the sequence $\mathbf{a} = (a_1, \ldots, a_k)$. Let τ_1 be defined by:

$$u \tau_1 v \Leftrightarrow u \tau_0 v \text{ or } u = v \text{ or } v \tau_0 u, \ (u, v \in Q^\wedge)$$

and τ be the transitive extension of τ_1, i. e. τ is defined by:

$$u \tau v \Leftrightarrow (\exists\, u_1, \ldots, u_p)\ u \tau_1 u_1 \tau_1 u_2 \tau_1 \ldots \tau_1 u_p \tau_1 v.$$

Then τ is a congruence on $Q^\wedge$, and $S = Q^\wedge / \tau$ is a periodic semigropp.

If $a \in Q$, $u \in Q^\wedge$ and $a \tau_0 u$, then there exist $c_0, \ldots, c_q, a_1, \ldots, a_k \in Q$, and $j \in \{0, \ldots, q+1\}$ such that:

$$a = c_0 \ldots c_{j-1} (a_1 \ldots a_k)^r \ c_j \ldots c_q,$$

$$u = c_0 \ldots c_{j-1} (a_1 \ldots a_k)^{r+mn} c_j \ldots c_q,$$

where r and m are the numbers associated to $\mathbf{a} = (a_1, \ldots, a_k)$.

From the first of the last two equations (by (1.2)) it follows that $q + rk \equiv 0$ (modn), i.e. $q + rk + kmn \equiv 0$ (modn), and this, by (1.1), implies that:

$$a = c_0 \ldots c_{j-1} (a_1 \ldots a_k)^r \ c_j \ldots c_q =$$

$$= [c_0 \ldots c_{j-1} (a_1 \ldots a_k)^r c_j \ldots c_q] = [c_0 \ldots c_{j-1} (a_1 \ldots a_k)^{r+mn} c_j \ldots c_q]$$

$$= c_0 \ldots c_{j-1} (a_1 \ldots a_k)^{r+mn} c_j \ldots c_q = u.$$

It can be shown in the same way that: $a \in Q, u \in Q^\wedge, \ u \tau_0 a \Rightarrow u = a$. Thus we have:

$$a \in Q,\ u \in Q^\wedge \ \Rightarrow \ (a \tau_1 u \Leftrightarrow a = u),$$

and this implies that:

$$a, b \in Q \Rightarrow (a \tau b \Leftrightarrow a = b). \tag{1.3}$$

From (1.3) it follows that the given n-semigroup Q can be embedded (as an n-subsemigroup) in the semigroup S, and this completes the proof.

We also note that the semigroup S satisfies the statement (1.2).

2. Most of the statements stated below are obvious corollaries from Theorem or its proof.

2.1. An n-semigroup Q is an n-subsemigroup of a periodic semigroup with a finite index iff:

$$(\exists\, r \geqslant 0)\ (\forall\, \mathbf{x})\ (\exists\, m > 0)\ (\forall\, \mathbf{y}, \mathbf{z})\ [\mathbf{y}\, \mathbf{x}^r\, \mathbf{z}] = [\mathbf{y}\, \mathbf{x}^{r+mn}\, \mathbf{z}]. \tag{2.1}$$

2.2. An n-semigroup Q is an n-subsemigroup of a periodic semigroup with a finite period iff:

$$(\exists\, m > 0)\ (\forall\, \mathbf{x})\ (\exists\, r \geqslant 0)\ (\forall\, \mathbf{y}, \mathbf{z})\ [\mathbf{y}\, \mathbf{x}^r\, \mathbf{z}] = [\mathbf{y}\, \mathbf{x}^{r+mn}\, \mathbf{z}]. \tag{2.2}$$

2.3. An semigroup Q is an n-subsemigroup of a semigroup with a finite index and a finite period iff:

$$(\exists\, r \geqslant 0,\ m > 0)\ (\forall\, \mathbf{x}, \mathbf{y}, \mathbf{z})\quad [\mathbf{y}\, \mathbf{x}^r\, \mathbf{z}] = [\mathbf{y}\, \mathbf{x}^{r+mn}\, \mathbf{z}]. \tag{2.3}$$

2.4. An n-semigroup Q is an n-subsemigroup of a periodic group iff:

$$(\forall\, \mathbf{x})\ (\exists\, m > 0)\ (\forall\, \mathbf{y}, \mathbf{z})\quad [\mathbf{y}\, \mathbf{z}] = [\mathbf{y}\, \mathbf{x}^{mn}\, \mathbf{z}]. \tag{2.4}$$

Then, Q is an n-group.

2.5. If $m > 0$ and $m \equiv 0$ (modn) then an n-semigroup is an n-subsemigroup of a group with an exponent a divisor of m iff

$$(\forall \mathbf{x}, \mathbf{y}, \mathbf{z}) \quad [\mathbf{y}\,\mathbf{z}] = [\mathbf{y}\,\mathbf{x}^m\,\mathbf{z}] \tag{2.5}$$

2.6. If Q is an n-subsemigroup of a periodic semigroup, then Q is a periodic n-semigroup.

2.7. The class of n-subsemigroups of periodic commutative semigroups is equal to the class of periodic commutative n-semigroups.

Proof. If Q is a periodic commutative n-semigroup, then (by 0.1) Q is an n-subsemigroup of a commutative semigroup S, and it can be assumed that S is generated by Q. Then S is periodic.

We notice also that the propositions **2.1** — **2.5** have obvious analogies in the case of commutativity.

Example. Let $A = \{a_k | \, k \in \mathbf{Z}\}$, $B = \{b_k | \, k \in \mathbf{Z}\}$ be two disjoint sets and $k \to a_k$, $k \to b_k$ be two injections from $\mathbf{Z}$ (the set of integers) into $G = A \cup B$. If we define a binary operation on G by:

$$a_i\, a_j = a_{i+j},\ a_i\, b_j = b_{i+j},\ b_i\, a_j = b_{i-j},\ b_i\, b_j = a_{i-j},$$

then we obtain a group in which B is a ternary subsemigroup. B is periodic, but G is not periodic. It can be easily shown that if B is a ternary subsemigroup of a semigroup S, then the inclusion mapping of B into G can be extended to an injective homomorphism of G into S, and this implies that S is not periodic.

REFERENCES

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n-ПОТПОЛУГРУПИ ОД ПЕРИОДИЧНИ ПОЛУГРУПИ

Резиме

Во трудов е даден опис на класата n-потполугрупи од периодични полугрупи.