SUBALGEBRAS OF SEMILATTICES

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Let $\mathcal{A}=(A, \Omega)$ be an algebra, S a semigroup and $\omega \mapsto \overline{\omega}$ a mapping of Ω into S, such that $A \subseteq S$ and

(1) $$\omega(a_1, \ldots, a_n) = \overline{\omega}\, a_1 \ldots a_n$$

for every n-ary operator $\omega \in \Omega$ and every $a_1, \ldots, a_n \in A$. $\mathcal{A}=(A, \Omega)$ is called a *subalgebra* of the semigroup S.

It is well known ([1] p. 185) that every algebra is a subalgebra of some semigroup. Subalgebras of commutative semigroups are characterized in [2].

In this paper we describe the class of algebras which are subalgebras of semilattices.

THEOREM. *An algebra $\mathcal{A}=(A, \Omega)$ is a subalgebra of some semilattice if and only if the following condition is satisfied:*

(*) *For every pair of terms t_1 and t_2, with the same sets of symbols, $t_1 = t_2$ is an identity in $\mathcal{A}$.*

Proof. It is obvious that the condition (*) is satisfied in every subalgebra of a semilattice.

Suppose that the condition (*) is satisfied in the algebra $\mathcal{A}=(A, \Omega)$. We are going to prove that this algebra is a subalgebra of a semilattice. We can assume that:

(i) A and Ω are disjoint sets;

(ii) Different operators from Ω define different operations in A;

(iii) Ω does not contain 0-ary operators, so that the operators, i.e. elements of Ω, are operations in A.

Le $K = A \cup \Omega$ and let $\mathcal{M}$ be the family of finite subsets of the set K (including the empty set). $\mathcal{M}$ is a semilattice with the set theoretical union as an operation. If we put $x = \{x\}$, for every $x \in K$, we have $K \subseteq \mathcal{M}$, so that $\mathcal{M}$ may be considered as a free semilattice with an identity on the base K.

Define in $\mathcal{M}$ a relation of „neighbourhood“ in the following way:

If $S, T \in \mathcal{M}$ and
$$S = S' \cup a, \quad T = S' \cup \{\omega, a_1, \ldots, a_n\},$$
where $a = \omega(a_1, \ldots, a_n)$ in the algebra $\mathcal{A}$, we say that (S, T) and (T, S) are two pairs of *neighbours generated by the operation* ω.

If there exists a sequence $S_0, S, \ldots, S_p \in \mathcal{M}$ such that $S = S_0$, $T = S_p$, $p \geq 0$ and (S_{i-1}, S_i) is a pair of neighbours for every $i \in \{1, 2, \ldots, p\}$, we say that S and T are *equivalent* and denote this $S \sim T$.

The relation $\sim$ is a congruence on the semilattice $\mathcal{M}$ and

(2) $$a = \omega(a_1, \ldots, a_n) \text{ (in } \mathcal{A}) \Rightarrow a \sim \{\omega, a_1, \ldots, a_n\}.$$

We are going to prove the following implication:

(3) $$a, b \in A \Rightarrow (a \sim b \Rightarrow a = b),$$

from which, because of (2), we come to the conclusion that the algebra $\mathcal{A}$ is a subalgebra of the semilattice $p = \mathcal{M}/\sim$.

To every element $S \in \mathcal{M}$ we correspond its „value“ $[S]$ in the following way: If one of the sets $S_\Omega = S \cap \Omega$, $S_A = S \cap A$ is empty, we put $[S] = S$.

Let
$$S_\Omega = \{\omega_1, \ldots, \omega_r\}, \ S_A = \{a_1, \ldots, a_s\}$$
be non-empty sets. If all the operations from S_Ω are unary we put
$$[S] = \{b_1, \ldots, b_s\}, \text{ where } b_\nu = \omega_1 \ldots \omega_r(a_\nu), \nu = 1, \ldots, s.$$

Finally, suppose that at least one of the operations from S_Ω for instance ω_r, is not unary. There exist positive integers i, j such that $\omega_1 \omega_2 \ldots \omega_r^i (a_1^j, a_2, \ldots, a_s)$ is a „continued product" and if a is the value of this product we put $[S]=a$. From the condition (*), it follows that $[S]$ does not depend on the quadruple ω_r, a_1, i, j.

Further on we denote $[S_1 \cup S_2 \cup \ldots \cup S_k]$ with $[S_1, S_2, \ldots, S_k]$.

The following lemmas are simple consequences from the condition (*) and the definition of the transformation [ ].

LEMMA 1. $[[S]]=[S]$.

LEMMA 2. *If* $S_A=T_A=\varnothing$, *then*

$$[S, T, U]=[S, [T, U]].$$

LEMMA 3. *If* $a=\omega(a_1, \ldots, a_n)$ in $\mathcal{A}$ *and* $[S, a]\in A$ *then*

$$[S, a]=[S, \omega, a_1, \ldots, a_n].$$

We are going to prove the following

LEMMA 4. *Let* $a=S_0, S_1, \ldots, S_p$ *be a sequence of elements from* $\mathcal{M}$, *where* $a\in A$, *and* (S_i, S_{i+1}) *is a pair of neighbours for every* $i\in\{0, 1, \ldots, p-1\}$. *If* ω_i *is the generating operation for the pair* (S_i, S_{i+1}) *then*

$$[\omega_\nu, a]=[\omega_1, \ldots, \omega_p, S_\nu]=a$$

for every $\nu\in\{1, \ldots, p\}$.

Proof. If $p=1$ then we have

$$S_1=\{\omega_1, a_1, \ldots, a_n\} \text{ or } S_1=\{a, \omega_1, a_1, \ldots, a_n\}$$

where $\omega_1(a_1, \ldots, a_n)=a$. Then

$$[\omega_1, a]=\omega_1(a^n)=\omega_1((\omega_1(a_1, \ldots, a_n))^n)=\omega_1(a_1, \ldots, a_n)=a$$

$$[\omega_1, S_1]=\omega_1(a^{n-1}, \omega_1(a_1, \ldots, a_n))=\omega_1(a^n)=a, n\geqslant 2$$

and

$$[\omega_1, S_1]=[\omega_1, \omega_1, a_1]=\omega_1^2(a_1)=\omega_1(a)=a, \text{ for } n=1.$$

Assume that the assertion of lemma 4. for the sequence $a=S_0, S_1, \ldots, S_q$ holds. We have two possible cases:

$$\text{(I) } S_{q+1}=S' \cup \{\omega, a_1, \ldots, a_n\},\ S_q=S' \cup b,\ b=\omega(a_1, \ldots, a_n)$$

$$\text{(II) } S_{q+1}=S' \cup b,\ S_q=S' \cup \{\omega, a_1, \ldots, a_n\},\ b=\omega(a_1, \ldots, a_n).$$

For case (I) we have

$$[\omega_1, \ldots, \omega_q, \omega, S_{q+1}]=[\omega_1, \ldots, \omega_q, \omega, \omega, S', a_1, \ldots, a_n]=$$
$$=[\omega_1, \ldots, \omega_q, S', \omega, a_1, \ldots, a_n].$$

From lemma 3. and an inductive assumption it follows that

$$a=[\omega_1, \ldots, \omega_q, S_q]=[\omega_1, \ldots, \omega_q, S', b]=[\omega_1, \ldots, \omega_q, S', \omega, a_1, \ldots, a_n]=$$
$$=[\omega_1, \ldots, \omega_q, \omega, S_{q+1}].$$

From lemma 2. we have

$$[\omega, a]=[\omega, [\omega_1, \ldots, \omega_q, \omega, S_{q+1}]]=[\omega_1, \ldots, \omega_q, \omega, S_{q+1}]=a,$$

$$[\omega, \omega_1, \ldots, \omega_q, S_\nu]=[\omega, [\omega_1, \ldots, \omega_q, S_\nu]]=[\omega, a]=a$$

for $\nu=1, \ldots, q$.

Consider case (II).

$\omega\in\{\omega_1, \ldots, \omega_q\}$ so it follows that

$$[\omega, \omega_1, \ldots, \omega_q, S_{q+1}]=[\omega_1, \ldots, \omega_q, S_{q+1}].$$

By the inductive assumption we also have

$$[\omega_1, \ldots, \omega_q, S_q]=a.$$

If in the set $\{\omega_1 \ldots, \omega_q\}$ there is a non-unary operation, then $[\omega_1, \ldots \omega_q, S_{q+1}] \in A$, and from lemma 3. we have

$$[\omega_1, \ldots, \omega_q, S_{q+1}]=[\omega_1, \ldots, \omega_q, S,' b]=$$
$$=[\omega_1, \ldots, \omega_q, S', \omega, a_1, \ldots, a_n]=[\omega_1, \ldots, \omega_q, S_q,]=a.$$

If all the operations $\omega_1, \ldots, \omega_q$ are unary then in can be easily sean that $[\omega_1, \ldots, \omega_q, S_q]=a$ impies that $[\omega_1, \ldots \omega_q, S_{q+1}]=a$. *Q.E.D.*

Now, implications (3) can easily be verified. If $a, b \in A$ and $a \sim b$, then there exists a sequence $a = S_0, S_1, \dots, S_p = b$ such that (S_{i-1}, S_i) is a pair of neighbours for every $i \in \{1, \dots, p\}$. If $p = 0$ then $a = S_0 = b$. Let $p \geqslant 1$ and the pair (S_{i-1}, S_i) be generated by the operation ω_i. From lemma 4. it follows that

$$[\omega_1, a] = [w_2, a] = \dots = [\omega_p, a] = [\omega_1, \dots, \omega_p, b] = a.$$

Consider the sequence $b = S_p, \dots, S_1, S_0 = a$. We have

$$b = [\omega_1, \dots, \omega_p, a] = [\omega_1, \dots, \omega_{p-1}, [\omega_p, a]] = [\omega_1, \dots, \omega_{p-1}, a] = [\omega_1, a] = a.$$

By this we finish the proof of the theorem.

Notes.

1. If Ω does not contain unary operators and to every n-ary operation $\omega \in \Omega$ we correspond a binary operation ω' by $\omega'(x, y) = \omega(x^{n-1}, y)$, then we have the algebra (A, Ω') with binary operations, satisfying the condition (*) if and only if the algebra (A, Ω) satisfies the same condition. We also have that

$$\omega(x_1, \dots, x_n) = \omega'^{n-1}(x_1, \dots, x_n).$$

2. We call the algebra (A, Ω) Ω-semilattice if there exists the semilattice P such that $A \subseteq P$ and

(1') $$\omega(a_1, \dots, a_n) = a_1 \dots a_n$$

for every n-ary operation $\omega \in \Omega$ and $a_1, \dots, a_n \in A$. It can be easily verified that (A, Ω) is Ω-semilattice if and only if the following condition is satisfied:

(*') For every pair of terms t_1, t_2, with the same sets of variables $t_1 = t_2$ is an identity in (A, Ω).

Obviously, the condition (*') is necessary.

If (*') is satisfied in (A, Ω) then all the operators, of the same arity n, define equal n-ary operations, so we may take that for every n there is at most one operation in Ω. If there exists a unary operation $\omega \in \Omega$ we have that $\omega(x) = x$ for every $x \in A$. Let ω be an n-ary operation in Ω ($n \geqslant 2$), and a binary operation „ $\cdot$ " be defined by $x \cdot y = \omega(x^{n-1}, y)$. Then we obtain a semilattice $(A, .)$ such that

$$\rho(x_1, \dots, x_m) = x_1 \dots x_m \text{ for every } m\text{-ary operation } \rho \in \Omega.$$

3. An algebra $\mathcal{A}$ satisfies condition (*) if and only if all the identities of the following forms hold in $\mathcal{A}$:

3.1 $\omega x_1 \dots x_n = \omega x_{i_1} x_{i_2} \dots x_{i_n}$;

3.2 $\omega \rho x_1 \dots x_m = \rho \omega x_1 \dots x_m$;

3.3 $\omega \rho x_1 \dots x_m = \omega x_1 \dots x_{i-1} \rho x_i \dots x_m$;

3.4 $\omega_1^{r_1} \dots \omega_p^{r_p} x_1^{\alpha_1} \dots x_q^{\alpha_q} = \omega_1^{s_1} \dots \omega_p^{s_p} x_1^{\beta_1} \dots x_q^{\beta_q}$.

$\mathcal{A}$ satisfies (*') if it satisfies all the identities 3.1, 3.2, 3.3 and 3.4', where

3.4' $\omega_1 \dots \omega_r x_1^{\alpha_1} \dots x_q^{\alpha_q} = \rho_1 \dots \rho_s x_1^{\beta_1} \dots x_q^{\beta_q}$.

($\omega, \rho, \omega_\nu, \rho_\nu$ are arbitrary elements of Ω; $i_1, \dots i_n$ is a permutation of $1, \dots, n$ and $n, m, r_\nu, s_\nu, \alpha_\nu, \beta_\nu, r, s$ are positive integers such that both the hand sides of the corresponding identities are Ω-terms).

REFERENCES

[1] P. M. Cohn, *Universal algebra*, New York, 1965.

[2] Ю. К. Ребане, *О предсшавлении универсальних алгебр в коммушашивных йолугрупах*, Сиб. Мат. Жур. 7 (1966), 878—885.

PODALGEBRE POLUMREŽE

REZIME

Neka je $\mathcal{A} = (A, \Omega)$ algebra, S polugrupa i $\omega \mapsto \overline{\omega}$ preslikavanje Ω u S tako da je $A \subseteq S$ i

$$\omega(a_1, \dots, a_n) = \overline{\omega} a_1, \dots, a_n$$

za svaki n-arni operator $\omega \in \Omega$ i $a_1, \dots, a_n \in A$. Kažemo da je $\mathcal{A}$ *podalgebra* polugrupe S.

U radu je dat opis klase podalgebri polumreža. Važi sledeća

TEOREMA. *Algebra $\mathcal{A} = (A, \Omega)$ je podalgebra neke polumreže ako i samo ako zadovoljava sledeći uslov*:

(*) *Za proizvoljne terme t_1 i t_2, sa jednakim skupovima simbola u algebri $\mathcal{A}$ je zadovoljen identitet $t_1 = t_2$.*