MULTIQUASIGROUPS AND SOME RELATED STRUCTURES

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The purpose of this paper is to show that the class of multiquasigroups is a convenient extension of the class of quasigroups. In the first part of the paper we give four interpretations of the notion of an $[n, m]$--quasigroup: (i) as a structure with a „vector valued" operation, (ii) as an algebra with a strongly orthogonal system of quasigroups, (iii) as an algebra with an orthogonal system of operations, and (iv) as a structure with a finitary relation. In the second part of the paper we show that on each (nontrivial) $[n, m]$-quasigroup it can be constructed an n-dimensional $n + m$-net, and conversely, each n-dimensional $n + m$-net can be coordinatized by an $[n, m]$-quasigroup. Partial multiquasigroups are considered in the third part of the paper, and it is shown that every partial $[n, m]$-quasigroup can be embedded in an $[n, m]$-quasigroup.

1. Let Q be a nonempty set, n and m positive integers, and $f: (x_1, \dots, x_n) \mapsto f(x_1, \dots, x_n)$ a mapping from Q^n into Q^m. Then we say that $Q(f)$ is an $[n, m]$-groupoid, and the n-ary operations $f_1, f_2, \dots, f_m$ defined by:

$$f(x_1, \dots, x_n) = (y_1, \dots, y_m) \Leftrightarrow (\forall\, i \in N_m)\ y_i = f_i(x_1, \dots, x_n),$$

are called the component operations of f and this is denoted by $f = (f_1, \dots, f_m)$.

An $[n, m]$-groupoid $Q(f)$ is said to be an $[n, m]$-quasigroup iff the following statement is satisfied:

(A). For each „vector" $(a_1, \dots, a_n) \in Q^n$ and each injection φ from $N_n = \{1, 2, \dots, n\}$ into N_{n+m}, there exists a unique vector $(b_1, \dots, b_{n+m}) \in Q^{n+m}$ such that $b_{\varphi(1)} = a_1, \dots, b_{\varphi(n)} = a_n$ and:

$$f(b_1, \dots, b_n) = (b_{n+1}, \dots, b_{n+m}). \tag{1}$$

It is clear that the following proposition is satisfied.

1.1. An $[n, m]$-groupoid $Q(f)$ is an $[n, m]$-guasigroup iff the sequence $f_1, \dots, f_m$ of component operations of f satisfies the statement (A') which is obtained from (A) by replacing (1) by: $(\forall\, i \in N_m)\ f_i(b_1, \dots, b_n) = b_{n+i}$. □

A sequence $f_1, \dots, f_m$ of n-ary operations on a set Q is said to be a strongly orthogonal system of operations if it satisfies the statement (A'). And, a sequence $g_1, \dots, g_{n+m}$ of n-ary operations on a set Q is called orthogonal if the following statement is satisfied.

(B). For each $(a_1, \dots, a_n) \in Q^n$ and each injection $\varphi: N_n \to N_{n+m}$ there exists a unique vector $(c_1, \dots, c_n) \in Q^n$ such that:

$$(\forall\, i \in N_n)\ g_{\varphi(i)}(c_1, \dots, c_n) = a_i.$$

The following proposition shows that there is an equivalence between the notions of orthogonal system of operations and $[n, m]$-quasigroups.

1.2. An $[n, m]$-groupoid $Q(f)$ is an $[n, m]$-quasigroup iff there exists an orthogonal system of n-ary operations $g_1, \dots, g_{n+m}$ such that:

$$f(x_1, \dots, x_n) = (x_{n+1}, \dots, x_{n+m}) \Leftrightarrow$$
$$(\exists\, t_1, \dots, t_n \in Q)\ (\forall\, i \in N_{n+m})\ x_i = g_i(t_1, \dots, t_n). \tag{2}$$

Proof. If $Q(f)$ is an $[n, m]$-quasigroup and if $g_1, \dots, g_{n+m}$ are defined by

$$f(x_1, \dots, x_n) = (x_{n+1}, \dots, x_{n+m}) \Leftrightarrow (\forall\, i \in N_{n+m})\ x_i = g_i(x_1, \dots, x_n),$$

then an orthogonal system of n-ary operations $g_1, \dots, g_{n+m}$ is obtained.

And conversely, if $g_1, \ldots, g_{n+m}$ is an orthogonal system of n-ary operations on Q, and if the $[n, m]$-groupoid $Q(f)$ is defined by (2), then $Q(f)$ is an $[n, m]$-quasigroup. □

As a consequence from **1.1** and **1.2** (or directly) we obtain the following connection between orthogonal and strongly orthogonal systems of operations.

1.3. A sequence of n-ary operations $f_1, \ldots, f_m$ on a set Q is a strongly orthogonal system iff the sequence $g_1, \ldots, g_n, f_1, \ldots, f_m$ is an orthogonal system, where $g_1, \ldots, g_n$ are defined by: $(\forall\, i \in N_n)\;\; g_i(x_1, \ldots, x_n) = x_i$. □

It is easy to see that in a strongly orthogonal system of n-ary operations on a set Q all operations are n-quasigroups.

An orthogonal system of n-quasigroups for $n = 2$ is a strongly orthogonal system, but for $n > 2$ a system of n-quasigroups which is an orthogonal system need not be a strongly orthogonal system.

An $n + m$-ary relation $\rho \in Q^{n+m}$ is called an $[n\ m]$-quasigroup relation if it satisfies the statement (A") obtained form (A) by replacing (1) by: $\rho\,(b_1, \ldots, b_{n+m})$.

The proof of the following proposition is also clear.

1.4. $Q(f)$ is an $[n, m]$-quasigroup iff the relation ρ defined by:

$$\rho\,(x_1, \ldots, x_{n+m}) \Leftrightarrow f(x_1, \ldots, x_n) = (x_{n+1}, \ldots, x_{n+m})$$

is an $[n, m]$-quasigroup relation. □

Thus, we obtained four interpretations of the notion of $[n, m]$-quasigroup. Further on we shall use mainly the last interpretation, i.e. by an "$[n, m]$-quasigroup" we shall mean a structure $Q(\rho)$ where ρ is an $[n, m]$-quasigroup relation. Then, we shall sometimes say that $Q(\rho)$ is a „multiquasigroup", if it is not necessary to emphasize n and m.

The proofs of the following properties are straightforward.

1.5. Let $\rho \subseteq Q^{n+m}$ and ψ be a permutation of N_{n+m}. Then $Q(\rho)$ is an $[n, m]$-quasigroup iff $Q(\rho_\psi)$ is an $[n, m]$-quasigroup, where:

$$\rho_\psi(x_1, \ldots, x_{n+m}) \Leftrightarrow \rho\,(x_{\psi(1)}, \ldots\ x_{\psi(n+m)}). \;\square$$

($Q(\rho_\psi)$ is called ψ parastroph of the multiquasigroup $Q(\rho)$.)

1.6. Let $\rho \subseteq Q^{n+m}$ and let $\xi = (\xi_1, \ldots, \xi_{n+m})$ be a sequence of permutations of Q. Then $Q(\rho)$ is an $[n, m]$-quasigroup iff $Q(\rho^\xi)$ is an $[n, m]$-quasigroup, where:

$$\rho^\xi(x_1, \ldots, x_{n+m}) \Leftrightarrow \rho\;(\xi_1\,(x_1), \ldots, \xi_{n+m}\,(x_{n+m})). \;\square$$

($Q(\rho^\xi)$ is called ξ-isotope of $Q(\rho)$.)

1.7. Let $Q(\rho)$ be an $[n, m]$-quasigroup, $a_1, \ldots, a_k \in Q$, $k < n$, and φ an injection form Q^k into Q^n. Then $Q(\rho')$ is also an $[n-k, m]$-quasigroup, where:

$$\rho'\,(x_1, \ldots, x_{n+m-k}) \Leftrightarrow \rho\,(x_1, \ldots, x_{n+m}) \wedge (\forall\, i \in N_k)\; x_{\varphi(i)} = a_i. \;\square$$

1.8. An $[n, 1]$-groupoid $Q(f)$ is an $[n, 1]$-quasigroup iff $Q(f)$ is an n-quasigroup. □

1.9. $Q(\rho)$ is a $[1, m]$-quasigroup iff there is a sequence $\xi_1, \ldots\ \xi_m$ of permutations of Q such that:

$$\rho\,(x, x_1, \ldots, x_m) \Leftrightarrow (\forall\, i \in N_m)\;\; x_i = \xi_i\,(x). \;\square$$

1.10. If $|Q| = 1$, and $\rho = Q^{n+m}$, then $Q(\rho)$ is an $[n, m]$-quasigroup.

An $[n, m]$-quasigroup $Q(\rho)$ is called nontrivial if $|Q| \geqslant 2$, $n \geqslant 2$, $m \geqslant 1$.

We remark that:

(i) The assumption that m and n are positive integers may be omited, and then we would obtain that there exist only trivial $[0, m]$-quasigroups, and $[n, 0]$-quasigroups. Namely, $Q(\rho)$ is an $[n, 0]$-quasigroup iff $\rho = Q^n$, and $Q(\rho)$ is a $[0, m]$-quasigroup iff $|\rho| = 1$.

(ii) The notion of an $[n, m]$-loop can be defined in a usual way, but it is easy to see that proper multiloops do not exist. We do not see any convenient definition of a proper multigroup.

2. Let $\mathbf{P}$ and $\mathbf{B}$ be two nonempty sets, $\mathbf{B} = \mathbf{B}_1 \cup \ldots \cup \mathbf{B}_{n+m}$ a partition of $\mathbf{B}$, where $n \geqslant 2$, $m \geqslant 1$, and I is a subset of $\mathbf{P} \times \mathbf{B}$ (the elements of $\mathbf{P}$ are called „points" and those of $\mathbf{B}$ „blocks".) The structure $(\mathbf{P}; \mathbf{B}_1, \ldots, \mathbf{B}_{n+m}; I)$ is called an n-dimensional $n+m$-net (or simply: an $[n, n+m]$-net) if the following statements are satisfied.

(i) If $p \in \mathbf{P}$ then there exists exactly one sequence $B_1, \ldots, B_{n+m} \in \mathbf{B}$ such that pIB_s, $B_s \in \mathbf{B}_s$, for all $s \in N_{n+m}$.

(ii) If $\varphi: N_n \to N_{n+m}$ is an injection and $B_s \in \mathbf{B}_{\varphi(s)}$ then there exists exactly one $p \in \mathbf{P}$ such that $pI\,\mathbf{B}_s$ for all $s \in N_n$.

We shall show that there exists an equivalence between the theory of $[n, n+m]$-nets and $[n, m]$-quasigroups.

2.1. Every nontrivial $[n, m]$-quasigroup induces an $[n, n+m]$-net.

Proof. Let $Q(\rho)$ be a nontrivial $[n, m]$-quasigroup. Define a set of „points" by:

$$\mathbf{P} = \{(x_1, \ldots, x_{n+m}) \mid \rho(x_1, \ldots, x_{n+m})\}.$$

If $x \in Q$ and $s \in N_{n+m}$, then: $B_s^x = \{(x_1, \ldots, x_{n+m}) \in \mathbf{P} \mid x_s = x\}$ is called a „block". And, $\mathbf{B} = \{B_s^x \mid s \in N_{n+m}, x \in Q\}$ is the set of all blocks. Further, let $\mathbf{B}_1, \ldots, \mathbf{B}_{n+m}$ be defined by: $\mathbf{B}_s = \{B_s^x \mid x \in Q\}$.
Clearly, $\mathbf{B}$ si a disjoint union of $\mathbf{B}_1, \ldots, \mathbf{B}_{n+m}$.

It is easy to see that $(\mathbf{P}; \mathbf{B}_1, \ldots, \mathbf{B}_{n+m}; I)$ is an $[n, n+m]$-net, where $pIB_s^x \Leftrightarrow p \in B_s^x$. (We say that this net is induced by the given multiquasigroup.) □

2.2. Every $[n, n+m]$-net induces an $[n, m]$-quasigroup.

Proof. Let $(\mathbf{P}; \mathbf{B}_1, \ldots, \mathbf{B}_{n+m}; I)$ be an $[n, n+m]$-net.

We shall show that all the sets $\mathbf{B}_1, \ldots, \mathbf{B}_{n+m}$ have the same cardinal number.

First we note that (i) and $P \neq \emptyset$ imply that all the classes of blocks $\mathbf{B}_1, \ldots, \mathbf{B}_{n+m}$ are nonempty.

Let $r, s \in N_{n+m}$, $r, s \notin \{i_2, \ldots, i_n\}$, $1 \leqslant i_2 < \ldots < i_n < n+m$, and choose $B_\nu \in \mathbf{B}_{i_\nu}$ in an arbitrary way. If $B \in \mathbf{B}_r$, then by (ii) there exists exactly one point p such that pIB and pIB_ν for each $\nu \in \{2, \ldots, n\}$. By (i) there exists exactly one $B' \in B_s$ such that pIB'. This implies that a mapping ψ_{sr}: $B \mapsto B'$ of $\mathbf{B}_r$ into B_s is defined. In the same manner we define a mapping $\psi_{rs}: \mathbf{B}_s \to \mathbf{B}_r$. It is easy to see that $\psi_{rs}\psi_{sr} = 1_{B_r}$, $\psi_{sr}\psi_{rs} = 1_{B_s}$, and this implies that $\psi_{rs} = (\psi_{sr})^{-1}$ is a bijection.

Let Q be a set and φ_i: $Q \to \mathbf{B}_i$ a bijection for every $i \in N_{n+m}$. We define an $n+m$-ary relation ρ in Q by:

$$\rho(x_1, \ldots, x_{n+m}) \Leftrightarrow (\exists p \in \mathbf{P})\ (\forall i \in N_{n+m})\ p\,I\psi_i(x_i).$$

It can be easily seen that $Q(\rho)$ is an $[n, m]$-quasigroup, and that the $[n, n+m]$-net induced by $Q(\rho)$ is isomorphic to the given $[n, n+m]$-net. □

2.3. If $Q(\rho)$ and $Q'(\rho')$ are two $[n, m]$-quasigroups induced by an

$[n, n+m]$-net, then they are isotopic.

Proof. Assume that $(\mathbf{P}; \mathbf{B}_1, \ldots, \mathbf{B}_{n+m}; I)$ is an $[n, n+m]$-net, $\varphi_i: Q \to B_i$, $\varphi_i': Q' \to B_i$ are bijections for each $i \in N_{n+m}$, and $Q(\rho)$, $Q'(\rho')$ are the $[n, m]$-quasigroups defined as in the proof of 2.2.

If the sequence of bijections $\psi_1: Q \to Q', \ldots, \psi_{n+m}: Q \to Q'$ is defined by $\psi_i = \varphi_i'^{-1}\varphi_i$ then we obtain an isotopy from $Q(\rho)$ into $Q'(\rho')$. □

3. A substructre of an $[n, m]$-quasigroup (in general) is not an $[n, m]$-quasigroup, but it is a partial $[n, m]$-quasigroup according to the following definition.

If $\rho \subseteq Q^{n+m}$ is an $n+m$-ary relation on a nonempty set Q, then the structure $Q(\rho)$ is called a partial $[n, m]$-quasigroup if the following condition is satisfied.

(C) Let $\varphi: N_n \to N_{n+m}$ be an injection. If $\rho(x_1, \ldots, x_{n+m})$, $\rho(y_1, \ldots, y_{n+m})$ and $(\forall\, i \in N_n)\ x_{\varphi(i)} = y_{\varphi(i)}$ then $(\forall\, j \in N_{n+m})\ x_j = y_j$.

Clearly:

3.1. Every $[n, m]$-quasigroup is a partial $[n, m]$-quasigroup, and the class of partial $[n, m]$-quasigroups is hereditary.

Now, we shall show that:

3.2. Every partial $[n, m]$-quasigroup $R(\rho)$ is a substructure of an $[n, m]$-quasigroup $R'(\rho')$.

Proof. Let $\varphi: N_n \to N_{n+m}$ be an injection, and D_R^φ the subset of R^n defined by:

$(a_1, \ldots, a_n) \in D_R^\varphi \Leftrightarrow (\exists\, b_1, \ldots, b_{n+m})\ [\rho(b_1 \ldots b_{n+m}) \wedge (\forall\, i \in N_n)\, a_i = b_{\varphi(i)}]$.

Denote R by R_0, and ρ by ρ_0. Assume that $R_k(\rho_k)$ is a partial $[n, m]$-quasigroup, and define $R_{k+1}(\rho_{k+1})$ in the following way.

Let $\mathbf{a} = (a_1, \ldots, a_n) \in R_k^n \setminus D_k^\varphi$[1]), where $\varphi: N_n \to N_{n+m}$ is an injection. Define a sequence $(1_{\mathbf{a}\varphi}, \ldots, (n+m)_{\mathbf{a}\varphi})$ in the following way

$$(\forall\, i \in N_n)\ i_{\mathbf{a}\varphi} = a_i \ \text{ and } (\mathbf{a}, D_k^\varphi) = \{j_{\mathbf{a}\varphi} \mid j \notin \{\varphi(1), \ldots, \varphi(n)\}\}$$

consists of m elements and it is disjoint with R_k; it is also assumed that:

$$(\mathbf{a}, D_k^\varphi) \cap (\mathbf{b}, D_k^\psi) \neq \varnothing \Leftrightarrow \mathbf{a} = \mathbf{b} \wedge \varphi = \psi.$$

Now, we define the structure $R_{k+1}(\rho_{k+1})$ by:

$$R_{k+1} = R_k \cup \bigcup_{\varphi, \mathbf{a}} (\mathbf{a}, D_k^\varphi), \quad \rho_{k+1} = \rho_k \cup \{1_{\mathbf{a}\varphi} \ldots (n+m)_{\mathbf{a}\varphi} \mid \mathbf{a} \in R_k^n \setminus D_k^\varphi, \varphi\}.$$

It can be easily seen that $R_{k+1}(\rho_{k+1})$ is a partial $[n, m]$-quasigroup.

Finaly, let $R'(\rho')$ be defined by: $R' = \bigcup_k R_k$, $\rho' = \bigcup_k \rho_k$.

The structure $R'(\rho')$ is a partial $[n, m]$-quasigroup, for it is the union of the chain $\{R_k(\rho_k)\ k = 1, 2, \ldots\}$ of partial $[n, m]$-quasigroups, such that $R_k^n \subseteq D_{k+1}^\varphi$ for each injection $\varphi: N_n \to N_{n+m}$, and this implies that $R'(\rho')$ is an $[n, m]$-quaisgroup.

It is natural to say that $R'(\rho')$ is the universal covering of $R(\rho)$. The universal covering $B'(\rho)$ of the partial $[n, m]$-quisigroup $B(\varnothing)$ is in fact the free $[n, m]$-quasigroup with a base B.

As consequences of 3.2 we obtain the following propositions.

3.3. If Q is an infinite set then there exists an $[n, m]$-quasigroup $Q(\rho)$. □

3.4. The free $[n, m]$-quasigroup with a finite (non-empty) base is countable and infinite. □

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[1] $D_k^\varphi = D_{R_k}^\varphi$

Making an obvious modification of the proof of 3.2, we obtain that the following statement is also satisfied.

3.5. Let $\varphi: N_n \to N_{n+m}$ be an injection, and $R(\rho)$ a partial $[n, m]$-quasigroup such that $D_R^{\varphi} \neq R^n$. There exists a partial $[n, m]$-quasigroup $Q(\rho)$ with the following properties:

(i) $Q(\rho)$ is an extension of $R(\rho)$;

(ii) $\psi: N_n \to N_{n+m}$ is an injection such that $D_Q^{\psi} = Q^n$ iff $\psi = \varphi$;

(iii) If R is infinite then $|R| = |Q|$.

Denote by $\Sigma_R^{n, m}$ the set of $n+m$-ary relations ρ on a set R such that $R(\rho)$ is a partial $[n, m]$-quasigroup. By an application of Zorn's lemma we obtain the following proposition.

3.6. Every relation $\rho \in \Sigma_R^{n, m}$ is contained in a maximal relation $\tau \in \Sigma_R^{n, m}$.

The following statements are also obvious.

3.7. If $\rho \in \Sigma_R^{n, m}$ and if $\varphi: N_n \to N_{n+m}$ is an injection such that D_ρ^{φ}[1] $= R^n$ then ρ is a maximal element in $\Sigma_R^{n, m}$.

3.8. If $n+m=n'+m'$ and $n \geqslant n'$, then $\Sigma_R^{n', m'} \subseteq \Sigma_R^{n, m}$.

Now, we shall show that every finitary relation on a set R induces a partial multiquasigroup.

3.9. If $\rho \subseteq R^k$ $(k \geqslant 0)$, then there exist n, m such that $k = n + m$ and $R(\rho)$ is a partial $[n, m]$-quasigroup.

Proof. If $\rho = \varnothing$ or $|\rho| = 1$, then $R(\rho)$ is a partial $[n, m]$-quasigroup for each pair of nonnegative integers n, m such that $n+m=k$. Let $|\rho| \geqslant 2$, and let d be the least positive integer such that there exist two vectors **a**, **b** with exactly d different components (in other words, d is the "code distance" of ρ). Then, $R(\rho)$ is a partial $[k-d+1,\ d-1]$-quasigroup. □

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МУЛТИКВАЗИГРУПИ И СТРУКТУРИ ПОВРЗАНИ СО НИВ

РЕЗИМЕ

Во работава се покажува дека мултиквазигрупите се подесно проширување на класата квазигрупи. Во првиот дел на работава се даваат четири интерпретации на поимот мултиквазигрупа: (i) како алгебра со една мултиоперација, (ii) како алгебра со силно ортогонален систем квазигрупи, (iii) како алгебра со еден ортогонален систем операции, и (iv) како една релациска структура. Во вториот дел се покажува дека на секоја $[n, m]$—квазигрупа може да се конструира n—димензионална $m+n$—решетка, а и обратно дека секоја таква решетка може да се координира со една $[n, m]$—квазигрупа. Делумни мултиквазигрупи се разгледуваат во третиот дел, а главниот резултат на овој дел е дека секоја делумна мултиквазигрупа може да се смести во мултиквазигрупа.

[1] $(a_1, \ldots, a_n) \in D_\rho^{\varphi} \Leftrightarrow (\exists\, b_1, \ldots, b_{n+m})\,[\rho\,(b_1, \ldots, b_{n+m}) \wedge (\forall\, i \in N_n)\; a_i = b_{\varphi(i)}]$