n-SUBSEMIGROUPS OF SOME COMMUTATIVE SEMIGROUPS

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A subset Q of a semigroup S is said to be an n-subsemigroup of S if: $a_1, \ldots, a_n \in Q \Rightarrow a_1 \ldots a_n \in Q$. Then, by $[x_1 \ldots x_n] = x_1 \ldots x_n$ is defined an associative n-ary operation [ ] on Q, i.e. (Q; [ ]) is an n-semigroup. If C is a class of semigroups, then the class of n-semigroups that are n-subsemigroups of C-semigroupes is denoted by $C(n)$. A variety of semigroups C is called an n-variety of semigroups iff $C(n)$ is a variety of n-semigroups. (Clearly, every variety of semigrops is a 2-variety.) The set of n-varieties of semigroups and its complement, in the set of varieties of semigroups, are infinite for any $n \geqslant 3$. ([1], [2], [3], [4]) It is also known that the set of n-varieties of commutative semigroups is infinite. ([2]) Here we show that if $n \geqslant 3$ then the set of varieties of commutative semigroups that are not n-varieties is also infinite.

Further on, by a semigroup (n-semigroup) we will mean a commutative semigroup (commutative n-semigroup).

A variety of semigroups is defined by a set of identities of the following form:

$$x_1^{i_1} x_2^{i_2} \ldots x_p^{i_p} = x_1^{j_1} x_2^{j_2} \ldots x_p^{j_p}, \qquad (*)$$

where: $x_1, x_2, \ldots$ are variables; i_ν, j_ν are nonnegative integers such that $(\Sigma\, i_\nu)(\Sigma\, j_\nu) > 0$. If $i_\nu = j_\nu$, for each ν, then (*) is called a trivial identity. (As usualy, variables will be also denoted by $x, y, z, \ldots$).

Let m, s, n, k be positive integers such that $s + 2 \leqslant m$, $m \neq 2s+1$, $m \neq 2s + 2$, $m \equiv 1\ (mod\ n-1)$ and $m + 2 \leqslant k$. Consider the following two identities:

$$x^s y^{m-s} = x^{s+2} y^{m-s-1}, \qquad (m, s)$$

$$x_1 x_2 \ldots x_k = x_1^2 x_2 \ldots x_k. \qquad (k)$$

Denote by $(m, s; k)$ the set of the given two identities, and by $(m, s; k)^n$ the set of identities (*) which are consequences of $(m, s; k)$ and the exponents satisfy the following condition:

$$\Sigma\, i_\nu \equiv \Sigma\, j_\nu \equiv 1 \ (\text{mod } n-1).$$

Every identity (*) which belongs to $(m, s; k)^n$ induces an identity

$$[x_1^{i_1} \cdots x_p^{i_p}] = [x_1^{j_1} \cdots x_p^{j_p}] \qquad [*]$$

of n-semigroups. We denote by $[m, s; k]^n$ the set of identities [*] such that (*) is in $(m, s; k)^n$.

It is clear that if $x^s y^{m-s} = x^i y^j$ is in $(m, s; k)^n$, then $i = s$, $j = m - s$, i.e. $[m, s; k]^n$ does not contain a nontrivial identity $[x^s y^{m-s}] = [x^i y^j]$.

Consider the variety $C^{(m, s; k)}$ of semigroups defined by $(m, s; k)$. We will show namely that $C^{(m, s; k)}$ is not an n-variety. To prove this statement it is enough to find an n-semigroup (Q; [ ]) which satisfies all identities belonging to $[m, s; k]^n$, but dces not belong to $C^{(m, s; k)}(n)$.

Let a, b, c be three different objects and let (Q; [ ]) be the n-semigroup with a presentation

$$< a, b, c;\ [a^{s+2} c^{m-s-2}] = [b^{s+2} c^{m-s-2}] > \qquad (**)$$

in the variety of n-semigroups defined by $[m, s; k]^n$. Let us give a more explicit construction of (Q; [ ]). First, let (F; [ ]) be the free n-semigroup with a basis $B = \{a, b, c\}$ in the variety defined by $[m, s; k]^n$. In other words F consists of all „commutative products of powers" $[a^i b^j c^p]$, such that: $i, j, p \geqslant 0$, $i + j + p \equiv 1 \ (\text{mod } n-1)$, and the equality

$$[a^j b^j c^p] = [a^{i'} b^{j'} c^{p'}]$$
holds in F iff the following identity
$$[x^i y^j z^p] = [x^{i'} y^{j'} z^{p'}]$$
is in $[m, s; k]^n$. The operation [ ] is defined in the usual way, i.e. by the following equation:
$$[[a^{i_1} b^{j_1} c^{p_1}] \dots [a^{i_n} b^{i_n} c^{p_n}]] = [a^{i_1 + \dots + i_n} b^{j_1 + \dots + j_n} c^{p_1 + \dots + p_n}].$$
Consider the minimal congruence $\approx$ on $(F; [\ ])$ such that
$$[a^{s+2} c^{m-s-2}] \approx [b^{s+2} c^{m-s-2}].$$
Namely $\approx$ is defined in the following way. If $u = a^j b^j c^k$ is such that $i + j + k \equiv 0 \ (mod\ n - 1)$, then:
$$[u\, a^{s+2} c^{m-s-2}] \sim [u\, b^{s+2} c^{m-s-2}] \text{ and } [u\, b^{s+2} c^{m-s-2}] \sim [u\, a^{s+2} c^{m-s-2}].$$
Now, $\approx$ is the transitive and reflexive extension of $\sim$, i.e. $u \approx v$ iff there exist $u_0, \dots, u_t \in F$ such that $u = u_0$, $v = v_t$, $t \geqslant 0$, and $u_{i-1} \sim u_i$ if $i \geqslant 1$. Then $(F/\approx; [\ ])$ is the desired n-semigroup $(Q; [\ ])$. We can assume that $a, b, c \in Q$.

Let us show that $[a^s c^{m-s}] \neq [b^s c^{m-s}]$ in $(Q; [\ ])$. Namely, we first conclude that if $[a^s c^{m-s}] = [a^i b^j c^p]$ in F, then $i = s$, $j = 0$, $p = m - s$. We also have that $[a^s c^{m-s}] \sim' [a^i b^j c^p]$, for any i, j, p, and therefore $[a^s c^{m-s}] \approx' [b^s c^{m-s}]$, i.e. $[a^s c^{m-s}] \neq [b^s c^{m-s}]$ in $(Q; [\ ])$.

Now, it is easy to show that $(Q; [\ ])$ does not belong to $C^{(m, s; k)}) (n)$. Namely, if $(Q; [\ ])$ were an n-subsemigroup of a semigroup $S \in C^{(n, s; k)}$, then we would have
$$\begin{aligned} a^s c^{m-s} &= a^{s+2} c^{m-s-1} = a^{s+2} c^{m-s-2} c \\ &= b^{s+2} c^{m-s-2} c = b^{s+2} c^{m-s-1} = b^s c^{m-s} \end{aligned}$$
in S, and this would imply the equality $[a^s c^{m-s}] = [b^s c^{m-s}]$ in $(Q; [\ ])$.

Clearly, if $m + 2 \leqslant k' < k''$ then $C^{(m, s; k')}$ is a proper subvariety of $C^{(m, s; k'')}$, and thus if s and m are fixed positive integers such that $s + 2 \leqslant m$, $m \neq 2s + 1$, $m \neq 2s + 2$, $m \equiv 1 \ (\mathrm{mod}\ n - 1)$ then we have an infinite set of varieties $\{C^{(m, s; k)} \mid k \geqslant n + 2\}$ of commutative semigroups which are not n-varieties.

Denote by $C^{(m, s)}$ ($C^{(k)}$) the variety of semigroups defined by the identity (m, s) ((k)). From the above considerations it follows that $C^{(m, s)}$ is not an n-variety. Namely, we notice again that there is not a nontrivial identity $[x^s y^{m-s}] = [x^i y^j]$ in $[m, s]^n$. And, the n-semigroup $(Q; [\ ])$ with a presentation (**) in the variety of n-semigroups defined by $[m, s]^n$ is not an n-subsemigroup of a semigroup belonging to $C^{(m, s)}$.

But, the variety $C^{(k)}$ is an n-variety for every pair of positive numbers n, k such that $n \geqslant 2$. (Namely, the assumption $k \geqslant n + 2$ is not necessary.). First we notice that (*) is a consequence of (k) if $i_\nu = j_\nu$ for each ν or $\Sigma\, i_\nu \geqslant k$, $\Sigma\, j_\nu \geqslant k$ and $i_\nu > 0 \Leftrightarrow j > 0$. This gives a complete description of $[k]^n$, as well.

Assume now that $(Q; [\ ])$ is an n-semigroup which satisfy any identity of $[k]^n$, i.e. each identity [*] such that
$$\Sigma\, j_\nu \equiv \Sigma\, j_\nu \equiv 1 \ (\mathrm{mod}\ n - 1), \quad \Sigma\, i_\nu \geqslant k, \Sigma\, j_\nu \geqslant k$$
and $i_\nu > 0 \Leftrightarrow j_\nu > 0$.

We have to show that $(Q; [\ ])$ is an n-subsemigroup of a semigroup belonging to $C^{(k)}$.

Let $n \geqslant k$, and let a binary operation $\cdot$ be defined on Q by:

$$x \cdot y = [x\, y^{n-1}].$$

Then it easy to see that $(Q; \cdot)$ is a semigroup in $C^{(k)}$, and moreover that

$$x_1 x_2 \ldots x_n = [x_1 x_2 \ldots x_n],$$

and therefore $(Q; [\ ])$ is an n-subsemigroup of $(Q; \cdot)$.

In the general case, we consider the semigroup S with a presentation

$$< Q; \{a = a_1 \ldots a_n \mid a = [a_1 \ldots a_n] \quad \text{in} \quad (Q; [\ ])\} >$$

in the variety $C^{(k)}$, and we have to show that:

$$a, b \in Q \Rightarrow (a = b \quad \text{in} \quad S \Rightarrow a = b \quad \text{in} \quad Q),$$

but here we will not give the complete proof of this statement.

REFERENCES

[1] Марковски С.: За дистрибутивните полугрупи, Год. збор. Матем. фак. Скопје 30 (1979), 15—27

[2] Марковски С.: За една класа полугрупи, Математички Билтен Кн. 1 (XXVII) 29—36

[3] Чупона Ѓ.: Полугрупи генерирани од асоцијативи, Год. зборн. ПМФ — Скопје Секц. А 15 (1964), 5—25

[4] Čupona G.: n-subsemigroups of semigroups satisfying the law $x^r = x^{r+m}$, Год. зборник Матем. фак. Скопје, 30 (1979), 5—14

n — ПОТПОЛУГРУПИ ОД НЕКОИ КОМУТАТИВНИ ПОЛУГРУПИ

Резиме

Во работава се покажува дека множеството многубразија M комутативни полугрупи, такви што $M(n)$ (т.е. класата од n-потполугрупи од M-полугрупи) е вистинско квазимногуобразие, е бесконечно, за секое $n > 3$.