REPRESENTATIONS OF UNARY ALGEBRAS IN UNARS

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A *unary F-algebra* is a universal algebra $\mathbf{A} = (A; F)$ with a carrier A on which each $f \in F$ induces a unary operation $a \mapsto a^f$. If $F = \{f\}$ is a one-element set, then $\mathbf{A} = (A; f)$ is called a *unar*, and then we usually write a' instead of a^f. If $(B; ')$ is a unar, then a mapping $(b, k) \mapsto b^k$ of $B \times N$ into B is defined by: $b^o = b$, $b^{k+1} = (b^k)'$, where $b \in B$, $k \in N$ (N is the set of non-negative integers). Let $l : f \mapsto |f|$ be a mapping of F into N, $\mathbf{A} = (A; F)$ be an F-algebra and $\mathbf{B} = (B; ')$ be a unar. A mapping $\varphi : A \to B$ is called an *l-homomorphism of* $\mathbf{A}$ *into* $\mathbf{B}$ if

$$\varphi(x^f) = (\varphi(x))^{|f|}$$

for any $x \in A$ and $f \in F$. If α is a cardinal number such that $\operatorname{Card} \varphi(A) \leqslant \alpha$ for every l-homomorphism $\varphi : A \to B$ and the equality $\alpha = \operatorname{Card} \psi(A)$ holds for at least one l-homomorphism $\psi : A \to B$, then we say that α is the *l-order* of $\mathbf{A}$, and we write $\| \mathbf{A} \|_l = \alpha$ or, simply, $\| \mathbf{A} \| = \alpha$. If $\| \mathbf{A} \| = 1$, then $\mathbf{A}$ is said to be *l-singular*.

Some properties concerning l-orders, or l-singularity of unary algebras are shown in this paper. Namely, we show that almost all the results obtained in [2] for semigroup orders of universal algebras have corresponding analogies for unary algebras.

1. SINGULAR UNARY ALGEBRAS

Consider first the case when $\operatorname{Card} F \geqslant 2$. Let f, g be two different elements of F such that $m = |f| \geqslant |g| = n$. Suppose that A is a non-empty set and e a fixed element of A. Let $\mathbf{A} = (A; F)$ be a unary F-algebra such that

$$(\forall x \in A) \quad x^f = x, \quad x^g = e.$$

If $\mathbf{B} = (B; ')$ is a unar and $\varphi : \mathbf{A} \to \mathbf{B}$ an l-homomorphism, then we have:

$$\varphi(x) = \varphi(x^f) = (\varphi(x))^m = ((\varphi(x))^n)^{m-n} = (\varphi(x^g))^{m-n} = (\varphi(e))^{m-n} =$$
$$= (\varphi(e^g))^{m-n} = ((\varphi(e))^n)^{m-n} = (\varphi(e))^m = \varphi(e^f) = \varphi(e),$$

for any $x \in A$. Therefore $\mathbf{A}$ is an l-singular algebra.

Thus we have proved the following proposition:

1.1. *If* $\operatorname{Card} F \geqslant 2$, *then any non-empty set* A *is the carrier of an* *l-singular algebra*. ∎

Assume again that $\operatorname{Card} F \geqslant 2$, and let f, g, m, n be as above. Let A be a non-empty set and e an object such that $e \notin A \times N$. Let $\mathbf{A}^*$ be an F-algebra with the carrier $A^* = \{e\} \cup A \times N$ such that:

(i) $e^g = e^f = e$,

(ii) $(x, k + 1)^g = e$, $(x, k + 1)^f = (x, k)$ for any $x \in A$, $k \in N$.

Let $\varphi : A^* \to B$ be an l-homomorphism from $\mathbf{A}^*$ into a unar $\mathbf{B}$. Then we have:

$$\varphi(x, k) = \varphi((x, k+1)^f) = (\varphi(x, k+1))^m = ((\varphi(x, k+1))^n)^{m-n} =$$
$$= ((\varphi((x, k+1)^g))^{m-n} = (\varphi(e))^{m-n} = ((\varphi(e))^n)^{m-n} =$$
$$= (\varphi(e))^m = \varphi(e^f) = \varphi(e),$$

for every $x \in A$, $k \in N$. This implies that $\mathbf{A}^*$ is an l-singular algebra.

Now, we can show the following proposition.

1.2. *If* $\operatorname{Card} F \geqslant 2$, *then every F-algebra is a subalgebra of an* *l-singular F-algebra*.

Namely, a unary F-algebra $\mathbf{A} = (A; F)$ can be embedded as a subalgebra in an algebra $\mathbf{A}^*$ defined as above. ∎

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It remains the case when $F = \{f\}$ is a one-element set. Then, if $|f| = n$, we say „an n-*singular unar*" instead of „an l-*singular unar*".

We have shown in [3] that if $(A; f)$ is a unar and $n \geqslant 1$, then there exists a unar $(B; ')$ such that $A \subseteq B$ and $a^f = a^n$, for any $a \in A$. This implies the following result:

1.3. *If $n \geqslant 1$, then a unar $(A; f)$ is n-singular iff* Card $A = 1$. ∎

We recall (see, for example, [5]) that if a relation $\sim$ is defined in a unar $(A; f)$ by

$$x \sim y \Leftrightarrow (\exists p, q \in N) \; x f^p = y f^q \,,$$

then a congruence is obtained, and if $(B; ')$ is the corresponding factor-unar, then we have: $b' = b$, for any $b \in B$. Then the canonical mapping

$$\text{nat}_\sim : a \mapsto b \quad (a \in b)$$

is a 0-homomorphism of $(A; f)$ in $(B; ')$. Assume now that φ is an arbitrary 0-*homomorphism* from $(A; f)$ into a unar $(C; *)$, i. e. $\varphi(a^f) = \varphi(a)$ for every $a \in A$. This implies that:

$$x \sim y \Rightarrow \varphi(x) = \varphi(y), \quad \text{i. e. } \sim \subseteq \ker \varphi.$$

A $\sim$-equivalence class is called a *connected class* of $(A; f)$ and the unar is *connected* iff there exists only one connected class.

Thus we have the following result:

1.4. *The 0-order of a unar is the number of its connected classes.* (*Therefore, a unar is 0-singular iff it is connected.*) ∎

As a corollary we obtain the following two propositions:

1.5. *Let A be a non-empty set and α a cardinal number such that $0 < \alpha \leqslant$ Card A. Then there is a unar $(A; f)$ with the 0-order α. Therefore, any non-empty set is the carrier of a 0-singular unar.*) ∎

1.6. *Let $\mathbf{B}$ be a subunar of a unar $\mathbf{A}$. If α is the 0-order of $\mathbf{A}$ and β is the 0-order of $\mathbf{B}$, then $\beta \leqslant \alpha$.* (*Thus, every subunar of a 0-singular unar is a 0-singular unar.*) ∎

From the above results it follows that neither of the propositions **1.1**, **1.2** hold for n-singular unars if $n > 0$. As concerns the 0-singularity, we have the same situation with **1.2**, but **1.1** is satisfied.

2. UNARY F-ALGEBRAS WITH ARBITRARY UNARY ORDERS

Let $\mathbf{A} = (A; F)$ be a unary F-algebra and let $l : F \to N$ be an arbitrary mapping. Denote by $F(A)$ the set $\{x^f \mid x \in A, \; f \in F\}$, and put $B = \{e\} \cup (A \setminus F(A))$, where $e \notin A \setminus F(A)$. If a unar $\mathbf{B} = (B; ')$ is defined by $(\forall x \in B) \, x' = e$ and $\varphi : A \to B$ is defined by

$$\varphi(x) = \begin{cases} x & \text{if} \quad x \in A \setminus F(A), \\ e & \text{if} \quad x \in F(A), \end{cases}$$

then an l-homomorphism from $\mathbf{A}$ into $\mathbf{B}$ is obtained. Thus:

2.1. *If $\mathbf{A} = (A; F)$ is an arbitrary F-algebra and $l : F \to N$ is an arbitrary mapping, then the following inequality is satisfied:*

$$\| \mathbf{A} \| \geqslant \text{Card} \, (A \setminus F(A)) + 1. \quad ∎ \tag{2.1}$$

As a consequence from (2.1) we obtain that:

2.2. *If $\mathbf{A} = (A; F)$ is l-singular for some l, then it is surjective, $F(A) = A$.* ∎

Let $\mathbf{A}$ be a subalgebra of a unary F-algebra $\mathbf{A}^*$ and let $\varphi : \mathbf{A}^* \to \mathbf{B}$ be such an l-homomorphism that Card $\varphi(A^*) = \| \mathbf{A}^* \|$. Then the restriction φ_A of φ on A is an l-homomorphism as well and this implies that:

$$\| \mathbf{A}^* \| = \text{Card} \, \varphi(A^*) = \text{Card} \, \varphi_A(A) + \text{Card} \, \varphi(A^* \setminus A)$$
$$\leqslant \| \mathbf{A} \| + \text{Card} \, (A^* \setminus A).$$

Therefore the following proposition holds:

2.3. *If* **A** *is a subalgebra of a unary F-algebra* $\mathbf{A^*} = (A^*; F)$, *then the following inequality is satisfied*:

$$\|\mathbf{A^*}\| \leqslant \|\mathbf{A}\| + \operatorname{Card}(A^* \setminus A) \qquad (2.2)$$

for any mapping $l : F \to N$. ∎

Now we will show that every F-algebra **A** is a subalgebra of an F-algebra **A*** such that the equality holds in (2.2).

2.4. *Let* **A** *be a unary F-algebra and* α *an arbitrary cardinal number. There is an F-algebra* $\mathbf{A^*} = (A^*; F)$ *such that* **A** *is a subalgebra of* **A*** *and the following equalities are satisfied*:

$$\alpha = \operatorname{Card}(A^* \setminus A), \quad \|\mathbf{A^*}\| = \|\mathbf{A}\| + \alpha.$$

Proof. We can assume that $\alpha > 0$, for if $\alpha = 0$, then there is nothing to prove. Let C be a set disjoint with A such that $e \in C$ and $\operatorname{Card} C = \alpha$. Let $A^* = A \cup C$ and let $\mathbf{A^*} = (A^*; F)$ be defined in the following way:

(i) **A** is a subalgebra of **A***;

(ii) $(\forall x \in C, f \in F) \quad x^f = e$.

Then, by **2.3**, we have: $\|\mathbf{A^*}\| \leqslant \|\mathbf{A}\| + \alpha$.

Let $\varphi : A \to B$ be an l-homomorphism such that $B \cap C = \emptyset$ and $\operatorname{Card} \varphi : (A) = \|\mathbf{A}\|$. Define a unar $\mathbf{B^*} = (B \cup C; {}')$ such that:

(iii) **B** is a subunar of **B***;

(iv) $(\forall x \in C) \quad x' = e$.

Extend the mapping φ to a mapping $\psi : A^* = A \cup C \to B \cup C = B^*$ by

$$\psi(x) = \begin{cases} \varphi(x) & \text{if} \quad x \in A \\ x & \text{if} \quad x \in C. \end{cases}$$

Then $\psi : \mathbf{A^*} \to \mathbf{B^*}$ is an l-homomorphism such that $\psi(A^*) = \psi(A) \cup C$ and therefore we obtain: $\|\mathbf{A^*}\| \geqslant \operatorname{Card} \psi(A^*) = \operatorname{Card} \varphi(A) + \alpha = \|\mathbf{A}\| + \alpha$. This, finally, implies that $\|\mathbf{A^*}\| = \|\mathbf{A}\| + \alpha$, which completes the proof. ∎

Further on the algebra **A*** obtained in the proof of the previous proposition will be denoted by $\mathbf{A}(C)$.

Now we can generalize the proposition **1.2**.

2.5. *If* $\alpha \, (\neq 0)$ *is a given cardinal and* $\operatorname{Card} F \geqslant 2$, *then every F-algebra* **A** *is a subalgebra of an F-algebra* **A**** *such that* $\|\mathbf{A^{**}}\| = \alpha$.

Proof. By **1.2**, **A** is a subalgebra of an l-singular algebra **A***. If $\alpha = 1$, then **A*** is the desired algebra, and thus we can assume that $\alpha > 1$. Let C be a non-empty set such that $A \cap C = \emptyset$ and $1 + \operatorname{Card} C = \alpha$. Then **A*** is a subalgebra of $\mathbf{A^*}(C)$ and by **2.4.** we have

$$\|\mathbf{A^*}(C)\| = \|\mathbf{A^*}\| + \operatorname{Card} C = 1 + \operatorname{Card} C = \alpha. \quad \blacksquare$$

The proposition **1.1** can be generalized as well. Assume that $\operatorname{Card} F \geqslant 2$ and that α is a given cardinal such that $1 \leqslant \alpha \leqslant \operatorname{Card} A$. Let $A = A^* \cup C$, $A^* \cap C = \emptyset$ and $1 + \operatorname{Card} C = \alpha$. By **1.1**, there is an l-singular algebra $\mathbf{A^*} = (A^*; F)$. If $\mathbf{A} = \mathbf{A^*}(C)$, then by **2.4** we have

$$\|\mathbf{A}\| = \|\mathbf{A^*}\| + \operatorname{Card} C = 1 + \operatorname{Card} C = \alpha.$$

Thus we have the following proposition:

2.6. *Let* $\operatorname{Card} F \geqslant 2$ *and let* A *be a non-empty set. If* α *is a cardinal number such that* $1 \leqslant \alpha \leqslant \operatorname{Card} A$, *then there is an F-algebra* $\mathbf{A} = (A; F)$ *such that* $\|\mathbf{A}\| = \alpha$. ∎

If $F = \{f\}$ is a one-element set and if $|f| = n \geqslant 1$, then neither of the propositions **2.5**, **2.6** hold, for then the n-order of a unar is the usual order of the unar. And, if $|f| = 0$, then by **1.5** and **1.6** the proposition **2.6** is satisfied as well, but **2.5** does not hold.

The l-order of an F-algebra is closely connected with the *l-universal unar* $\mathbf{A^\wedge}$ for the given algebra $\mathbf{A} = (A; F)$. Namely, $\mathbf{A^\wedge}$ is the unar with the following presentation:

$$< A; \{b = a^{|f|} \mid b = a^f \text{ in } \mathbf{A}\} > \qquad (2.3)$$

in the class of unars. A more explicit construction of $\mathbf{A}^{\wedge}$ can be found in [3]. If $a,\ b \in A$ and $a,\ b$ define the same element in $\mathbf{A}^{\wedge}$, then we write $a \approx b$ and we say that a and b are *equivalent*. Now we can state the following proposition:

2.7. *The relation $\approx$ is an equivalence on A and $\|\mathbf{A}\| = \mathrm{Card}\,(A/\approx)$.* (Therefore, $\mathbf{A}$ is l-singular iff $(\forall a, b \in A)\ a \approx b$.) ∎

We note that if $|f| = 0$ for each $f \in F$, then $\approx$ is the congruence on $\mathbf{A}$ generated by $\{(a, a^f) \mid a \in A\}$.

3. UNARY ORDERS OF J-UNARS

Let J be a subsemigroup of the additive semigroup of positive integers and let A be a non-empty set. If $(a,n) \mapsto an$ is a mapping from $A \times J$ into A satisfying the following condition

$$(\forall a \in A, \quad m, n \in J) \quad a(m + n) = (am)n, \qquad (3.1)$$

then we say that $\mathbf{A} = (A;J)$ is a *J-unar*. (In other words, a J-unar is a *right J-system* [1; 11. 1].) A J-unar can be also considered as a unary F-algebra, where

$$F = \{f_n \mid n \in J\} \text{ and } (\forall a \in A, \quad n \in J)\ a^{f_n} = an.$$

If we define a mapping $l : F \to N$ by $l(f_n) = n$, we can speak of the notion of the *l*-order of a J-unar. In this case we will say „unary order of A“ instead of „*l*-order of $\mathbf{A}$“; and the meaning of the notion „a singular J-unar“ will be clear.

We need some results on additive semigroups of positive integers.

3.1. *Let J be an additive semigroup of positive integers.*

(i) There exists a uniquely determined minimal generating subset $K = \{n_1, \ldots, n_k\}$ of J, which is called the basis of J.

(ii) If d is the largest common divisor of the numbers in K, then there exists a $t \in N$ such that $t + \nu d \in J$, for any $\nu \geqslant 0$. (If t_0 is the minimal number with that property, then the set $R(J) = \{t_0 + \nu d \mid \nu \in N\}$ is called the regular part of J. The basis of $R(J)$ will be denoted by $P = \{m_1, \ldots, m_p\}$.) ([4]) ∎

The universal unar $\mathbf{A}^{\wedge}$ for a J-unar $\mathbf{A}$ is defined as in the previous section. Therefore $\mathrm{Card}\,(A/\approx)$ is the unary order of the J-unar $\mathbf{A}$.

The following two results are proved in [3] :

3.2. *If $a, b \in A$ and $m \in R(J)$, then*

$$a \approx b \Rightarrow am = bm. \qquad (3.2)$$

3.3. *If a J-unar $(A;J)$ is surjective, i.e. $AJ = A$, then $An = A$ for any $n \in J$.* ∎

Now it is easy to show that every singular J-unar is trivial. Namely, if $(A;J)$ is a singular J-unar, then by **2.2** it is surjective, and by **3.3** we have $An = A$ for any $n \in J$. The singularity also implies that $a \approx b$ for any $a, b \in A$, and this by **3.2** implies that $am = bm$ for any $m \in R(J)$; thus, if $m \in R(J)$, the mapping $x \mapsto xm$ is a constant; on the other hand we have $Am = A$, and therefore we obtain that $\mathrm{Card}\,A = 1$. Thus we have proved the following proposition:

3.4. *A J-unar $(A;J)$ is singular iff $\mathrm{Card}\,A = 1$.* ∎

Some connections between the unar order of a J-unar $(A;J)$ and $\mathrm{Card}\,A$ will be established below.

Let $(A;J)$ be a J-unar and let $K, R(J), P$ be as in **3.1** and $Q = J \setminus R(J)$, $\mathrm{Card}\,Q = q$. Let α be the unary order of the given J-unar and B, C, A' be subsets of A defined as follows:

$$B = AP, \quad C = AJ \setminus B, \quad A' = A \setminus AJ.$$

If $a, b \in A$ are such that $a \approx b$, then for each $m \in P$ we have $am = bm$, and this implies that Card $Am \leqslant \alpha$, i. e. Card $B \leqslant \alpha p$. By **2.1** we have that Card $A' \leqslant \alpha - 1$. If $c \in C$ and if n is the maximal number of $J \setminus R(J)$ such that $c \in An$, then there is an element $a \in A'$ such that $c = an$. This implies that Card $C \leqslant q(\alpha - 1)$. Finally we obtain the following relation:

$$\text{Card } A = \text{Card } A' + \text{Card } B + \text{Card } C \tag{3.3}$$
$$\leqslant (\alpha - 1) + q(\alpha - 1) + p\alpha = \alpha(1 + p + q) - (q + 1).$$

The following propositions are obvious corollaries of (3.3).

3.5. *The unary order of an infinite J-unar* $(A; J)$ *is the cardinal of* A, *i. e. it is the usual order of the J-unar.* ∎

3.6. *A J-unar* $\mathbf{A} = (A; J)$ *has a finite unar order* α *iff* Card $A = \beta$ *is finite, and then we have:*

$$\beta \leqslant (p + q + 1)\alpha - (q + 1). \tag{3.3'}$$

Therefore the notion of unary order of *J*-unars could be of interest for finite *J*-unars only.

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Р Е З И М Е

ПРЕТСТАВУВАЊЕ УНАРНИ АЛГЕБРИ ВО УНАРИ

Една универзална алгебра $\mathbf{A} = (A; F)$ со носител A и множество F од унарни оператори, такви што секој $f \in F$ индуцира унарна операција $a \mapsto a^f$ на A, се вика унарна F-алгебра. Ако $F = \{f\}$, тогаш $\mathbf{A} = (A; f)$ се вика унар и, во тој случај, обично пишуваме a' наместо a^f.

Нека $\mathbf{A} = (A; F)$ е унарна алгебра, нека $l: f \mapsto |f|$ е пресликување од F во множеството N на природните броеви и нека $\mathbf{B} = (B; ')$ е унар. Едно пресликување $\varphi: A \to B$ се вика l-хомоморфизам од $\mathbf{A}$ во $\mathbf{B}$ ако $\varphi(x^f) = (\varphi(x))^{|f|}$ за секој $x \in A$ и $f \in F$. Ако α е кардинален број, таков што Card $\varphi(A) \leqslant \alpha$ за секој l-хомоморфизам φ од $\mathbf{A}$ во некој унар $\mathbf{B}$ и важи равенството $\alpha = $ Card $\psi(A)$ барем за еден l-хомоморфизам ψ од $\mathbf{A}$, тогаш за α велиме дека е *унарен l-ред* на $\mathbf{A}$ и пишуваме $\|\mathbf{A}\|_l = \alpha$ или, само, $\|\mathbf{A}\|$. Ако $\|\mathbf{A}\| = 1$, тогаш за F-алгебрата $\mathbf{A}$ велиме дека е *l-сингуларна*.

Во работава се испитуваат некои својства на унарните алгебри во врска со поимите l-ред и l-сингуларност. Се покажува, меѓу другото, дека скоро сите резултати, добиени во [2] за полугрупен ред на универзални алгебри, имаат соодветни аналогии за унарни алгебри.