ON SEMIGROUPS OF OPERATIONS

Прилози, мат.-техн. науки, МАНУ, Скопје, IV 1 (1983), 43–47

Dedicated to Prof. Blagoj S. Popov

Several abstract characterizations of algebras of operations can be found in [1]—[5] and many other papers. Here we consider two kinds of normed semigroups, and we show that every such semigroup is a subsemigroup of a corresponding semigroup of operations, and of a semigroup of sequences as well.

1. ADDITIVELY NORMED SEMIGROUPS

We say that $(S; + ; ||)$ is an additively normed semigroup (a.n.s.) iff $(S; +)$ is a semigroup (not necessarily commutative), and $x \mapsto |x|$ is a mapping from S into the set of positive integers, such that $|x + y| = |x| + |y| - 1$, for any $x, y \in S$.

Example 1.1. Let A be a non-empty set, and let $O_n(A)$ be the set of all n-ary operations on A, and $O(A) = \cup \{O_n(A) : n \geqslant 1\}$. If $f \in O_n(A)$, then we write $|f| = n$. Define a binary operation "+" on $O(A)$ by:

$$f + g(x_1, \dots x_{m+n-1}) = f(g(x_1, \dots, x_m), x_{m+1}, \dots, x_{m+n-1}),$$

where $|f| = n$, $|g| = m$. Then, $(O(A); + ; ||)$ is an a.n.s. An arbitrary a.n.s. $(S; +; ||)$ called an additively normed semigroup of operations (a.n.s.o.) if it is isomorphic to a subsemigroup of $(O(A); + ; ||)$, for some $A \neq \emptyset$.

Example 1. 2. Let $(B; \cdot)$ be a semigroup and $S(B) = \{(b_o, b_1, \dots, b_n) : b_v \in B, n \geqslant 1\}$ be the set of sequences on B with lengths $\geqslant 2$. Then, $(S(B); + ; ||)$ is an a.n.s., where:

$$a = (a_o, a_1, \dots, a_n), \quad b = (b_o, b_1, \dots, b_m),$$

implies that $|a| = n$, $|b| = m$, and:

$$a + b = (a_o b_o, b_1, \dots, b_{m-1}, b_m a_1, a_2, \dots, a_n).$$

The notion of an "additively normed semigroup of sequences" (a.n.s.s.) is clear.

THEOREM 1. *Every a.n.s. is an a.n.s.o. and an a.n.s.s. as well.*

Proof. Let $(S; + ; ||)$ be an a.n.s., and $a \in S$, $|a| = n$. Define an n-ary operation $\bar{a}$ on S in the following way:

$$(\forall x_1, \dots, x_n \in S)\, a(x_1, \dots, x_n) = a + x_1 + x_2 + \dots + x_n.$$

It is clear that $a \mapsto \bar{a}$ is a homomorphism from $(S; + ; ||)$ into $(O(S); + ; ||)$. This homomorphism is injective iff the following implication is satisfied:

$$a, b \in S, |a| = |b| = n \text{ and}$$
$$[(\forall x_1, \dots, x_n \in S)\, a + x_1 + \dots + x_n = b + x_1 + \dots + x_n] \Rightarrow a = b.$$

If this condition is not satisfied, then we can extend the given a.n.s. to an a.n.s. which does satisfy it. Namely, we can add a new element e as an identity and put $|e| = 1$. Thus, we obtain an a.n.s. $(S^e; + ; ||)$ which satisfies the above implication, and contains $(S; +; ||)$ as a subsemigroup.

This shows that every a.n.s. is an a.n.s.o.

(Certainly, the given proof is an obvious generalization of the well-known proof of the statement that every semigroup can be embedded into a semigroup of transformations.)

It remains to be shown that $(S; +; ||)$ is an a.n.s.s.

First we consider a subset $\hat{S}$ of $S \times N$ (N is the set of nonnegative integers) defined by:

$$\hat{S} = \{(a, i) \mid 0 \leqslant i \leqslant |a|\}.$$

(Instead of (a, i) we will write a_i.) Denote by $(a, b; c)$ the following set of "semigroup defining relations":

$$\{a_0 b_0 = c_0, b_1 = c_1, \dots, b_{m-1} = c_{m-1}, b_m a_1 = c_m, a_2 = c_{m+1}, \dots, a_n = c_{m+n-1}\},$$

where $|a| = n$, $|b| = m$, and $a + b = c$. Let $(B, \cdot)$ be the semigroup with the following presentation:

$$< \hat{S}; \cup \{(a, b; c) : a + b = c\} >$$

The mapping:

$$a \mapsto a^{\sim} = (a_0, a_1, \ldots, a_n), \text{ with } |a| = n,$$

is a homomorphism from $(S; + ; \|)$ into $(\mathsf{S}(B); +; \|)$.

Assume that $a^{\sim} = b^{\sim}$. Then we have $a_0 = b_0$ in B. But it is not difficult to show that if

$$a_0 = a_{i'}' a_{i''}'' \ldots a_{i^{(p)}}^{(p)} \text{ in } B$$

then

$$i' = i'' = \ldots = i^{(p)} = 0 \text{ and } a = a' + \ldots + a^{(p)} \text{ in } S.$$

Thus, $a_0 = b_0$ in $B \Rightarrow a = b$ in S, and therefore the mapping $a \mapsto a^{\sim}$ is an injective homomorphism.

This completes the proof of the Theorem.

2. MULTIPLICATIVELY NORMED SEMIGROUPS

By a multiplicatively normed semigroup (m.n.s.) we mean a structure $(S; * ; \|)$ such that $(S; *)$ is a semigroup and $x \mapsto |x|$ is a homomorphism from $(S;*)$ into the multiplicative semigroup of positive integers.

Example 2.1. If a binary operation * is defined on $\mathsf{O}(A)$ by:
$f*g(x_1, \ldots, x_{mn}) = f(g(x_1, \ldots, x_m), g(x_{m+1}, \ldots, x_{2m}), \ldots, g(\ldots, x_{mn}))$, where $A \neq \emptyset$, $|f| = n$, $|g| = m$, then a m.n.s. $(\mathsf{O}(A); * ; \|)$ is obtained.

Example 2.2. Let $(B, \cdot)$ be a semigroup and let an operation * be defined on $\mathsf{S}(B)$ as follows:

$$(a_0, a_1, \ldots, a_n) * (b_0, b_1, \ldots, b_m) =$$
$$= (a_0b_0, b_1, \ldots, b_{m-1}, b_m a_1 b_0, b_1, \ldots, b_{m-1}, \ldots, b_m a_{n-1} b_0, b_1, \ldots, b_{m-1}, b_m a_n).$$

Then $(\mathsf{S}(B); * ; \|)$ is an m.n.s.

The meanings of "m.n.s.o" and "m.n.s.s." are clear.

THEOREM 2. *Every m.n.s. is an m.n.s.o. and an m.n.s.s. as well.*

Proof. First we will show the second part of Theorem.

Let $(S; * ; \|)$ be an m.n.s. and let $\hat{S}$ be defined as in the proof of the second part of Theorem 1.

Define by $[a, b; c]$ the following set of semigroup defining relations:
$\{a_0 b_0 = c_0, b_1 = c_1, \ldots, b_{m-1} = c_{m-1}, b_m a_1 b_0 = c_m, b_1 = c_{m+1}, \ldots, b_{m-1} = c_{2m-1},$
$b_m a_2 b_0 = c_{2m}, \ldots, b_m a_{n-1} b_0 = c_{(n-1)m}, b_1 = c_{(n-1)m+1}, \ldots, b_{m-1} = c_{nm-1}, b_m a_n = c_{nm}\}$, where $a * b = c$, $|a| = n$, $|b| = m$. Consider the semigroup $(C; \cdot)$ determined by the following presentation:

$$< \hat{S}; \cup \{[a, b; c] : a * b = c\} >.$$

It can be shown in the same way as in the proof of the second part of Theorem 1, that if $a_0 = b_0$ in c then $a = b$ in S, and this implies that the mapping

$$a \mapsto (a_0, a_1, \ldots, a_n) \quad (|a| = n)$$

is an injective homomorphism from $(S; *; \|)$ into $(\mathsf{S}(C); *; \|)$. This proves that $(S; *; \|)$ is an m.n.s.s.

Now we will find an injective homomorphism from $(S; *; \|)$ into $(\mathsf{O}(C); *; \|)$, and thus the proof of Theorem will be complete.

Let $a \in S$, $|a| = n$, and let $\bar{\bar{a}} \in \mathsf{O}_n(C)$ be defined by:

$$\bar{\bar{a}}(x_1, x_2, \ldots, x_n) = a_0 x_1 a_1 x_2 \ldots a_{n-1} x_n a_n .$$

Clearly $a \mapsto \bar{\bar{a}}$ is a homomorphism from $(S; *; \|)$ into $(\mathsf{O}(C); *; \|)$. Assume that $a, b \in S$ are such that $\bar{\bar{a}} = \bar{\bar{b}}$. Then we have:

$$a_0 a_1 a_1 \ldots a_{n-1} a_1 a_n = \bar{\bar{a}}(a_1, \ldots, a_1) = \bar{\bar{b}}(a_1, \ldots, a_1) = b_0 a_1 b_1 \ldots b_{n-1} a_1 b_n$$

in C. But, it can easily be seen that if $a_0 a_1 u = b_0 a_1 v$ in C, then $a_0 = b_0$, and therefore $a = b$.

This completes the proof.

3. POSITION ALGEBRAS

The class of position algebras is introduced in [1], and position algebras of operations are considered first in [4]. (See also [3] and [5].) Namely, $(S; \{\overset{i}{+}: i \geqslant 1\}; ||)$ is a position algebra (p.a.) if $\{\overset{i}{+} : i \geqslant 1\}$ is a set of partial binary operations on S, and $x \mapsto |x|$ is a mapping from S into the set of positive integers, such that the following statements are satisfied:

(I) $a, b \in S, i \geqslant 1 \Rightarrow (a \overset{i}{+} b \in S \Leftrightarrow i \leqslant |a|)$;

(II) $1 \leqslant i \leqslant |a| \Rightarrow |a \overset{i}{+} b| = |a| + |b| - 1$;

(III) $1 \leqslant i \leqslant |a|, 1 \leqslant j \leqslant |b| \Rightarrow a \overset{i}{+} (b \overset{j}{+} c) = (a \overset{i}{+} b) \overset{i+j-1}{+} c$;

(IV) $1 \leqslant j < i \leqslant |a| \Rightarrow (a \overset{i}{+} b) \overset{j}{+} c = (a \overset{j}{+} c) \overset{i+|c|-1}{+} b$.

Example 3.1. Let $A \neq \emptyset$, $f \in \mathrm{O}_n(A)$, $g \in \mathrm{O}_m(A)$, and $1 \leqslant i \leqslant n$. Then $h = f \overset{i}{+} g \in \mathrm{O}_{m+n-1}(A)$ is defined by:

$$h(x_1, \ldots, x_{m+n-1}) = f(x_1, \ldots, x_{i-1}, g(x_i, \ldots, x_{m+i-1}), \ldots, x_{m+n-1}).$$

Thus we obtain a p.a. $(\mathrm{O}(A); \{\overset{i}{+} : i \geqslant 1\}; ||)$.

Example 3.2. Let $(B, \cdot)$ be a semigroup. A p.a. $(\mathrm{S}(B); \{\overset{i}{+} : i \geqslant 1\}; ||)$ can be defined as follows:

$$a \overset{i}{+} b = (a_0, \ldots, a_{i-2}, a_{i-1} b_0, b_1, \ldots, b_{m-1}, b_m a_i, a_{i+1}, \ldots, a_n),$$

where $a = (a_0, a_1, \ldots, a_n), b = (b_0, b_1, \ldots, b_m)$, and $1 \leqslant i \leqslant n$.

The meanings of "position algebras of operations" and of "position algebras of sequences" are clear.

THEOREM 3. *Every position algebra is a position algebra of operationss and the class of position algebras of sequences is a proper subclass of the clas, of position algebras.*

Proof. The first part of Theorem is shown in [5; p.p. 18, 23]. The second part of Theorem is a consequence of the fact that in the position algebras of sequences there hold some implications ("quasiidentities") which are not true in the class of all position algebras. For example, if $(S; \overset{i}{+} : i \geqslant 1\}; ||)$ is a position algebra of sequences, and if $a, b, c', c'', d', d'' \in S$ are such that $3 \leqslant |a| = |b|$ and $a \overset{1}{+} c' = b \overset{1}{+} c'', a \overset{3}{+} d' = b \overset{3}{+} d''$, then $a = b$. But, if A has at least two elements, this implication does not hold in $\mathrm{O}(A); \overset{i}{+}; i \geqslant 1\}; ||)$.

This completes the proof of Theorem 3.

Remark. It is clear that if $(S; \{\overset{i}{+} : i \geqslant 1\}; ||)$ is a position algebra, then $(S; +; ||)$ is an a.n.s. and $(S; *; ||)$ is an m.n.s., where:

$$a * b = (\ldots((a \overset{1}{+} b)(\overset{m+1}{+} b) \ldots)^{(n-1)m+1}_{+} b, \quad |a| = n, \quad |b| = m.$$

REFERENCES

[1] Belousov V. D.: *Balanced identitie s in algebras of quasigroups*, Aeq. Math. 8 (1972), 1—73.

[2] Schein, B. M. and Trohimenko V. S.: *Algebras of Multiplace Functions*, Semigroup Forum 17 (1979), 1—63.

[3] Белоусов В. Д.: *n-арные квазигруппы*, Кишинев 1972.

[4] Чупона Ѓ.: *За финитарните операции*, Год. зб. Филозоф. фак. Прир мат. оддел, Скопје, 12 (1959), 7—49.

[5] Чупона Ѓ.: *За алгебрите на сместувања*, Год. зб. Прир. мат. фак., Скопје, 20 (1970), 15—24.

ПОЛУГРУПИ НА ОПЕРАЦИИ

Резиме

Познати се поеке апстрактни карактеристики на алгебрите на операции. Овде се разгледуваат две класи нормирани полугрупи и се покажува дека секоја таква полугрупа може да се смести во соодветна полугрупа од операции.