VECTOR VALUED SUBGROUPOIDS OF SEMIGROUPS

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Abstract. If Q is a non-empty set, n, m two positive integers and f a mapping from Q^n to Q^m then we say that $(Q;f)$ is an (n,m)-groupoid, i.e. a vector valued groupoid. Let $(S;\cdot)$ be a semigroup such that $Q \subseteq S$ and

$$f(a_1, \dots, a_n) = (b_1, \dots, b_m) \Rightarrow a_1 \dots a_n = b_1 \dots b_m, \tag{0.1}$$

for any $a_\nu, b_\lambda \in Q$. Then $(Q;f)$ is said to be an (n,m)-subgroupoid, i.e. a vector valued subgroupoid, of $(S;\cdot)$. If for any a_ν, $b_\lambda \in Q$ we have:

$$a_1 \dots a_n = b_1 \dots b_m \Leftrightarrow f(a_1, \dots, a_n) = (b_1, \dots, b_m) \tag{0.1'}$$

then we say that $(Q;f)$ is a proper (n, m)-subgroupoid, i.e. a proper vector valued subgroupoid, of $(S;\cdot)$.

A complete description of vector valued subgroupoids of semigroups, and an almost complete description of proper vector valued subgroupoids of semigroups are given in this paper.

1. Below we will write v.v. groupoid instead of vector valued groupoid, and v.v.s. (p.v.v.s) instead of "vector valued subgroupoid" ("proper vector valued subgroupoid"); moreover, instead of v.v.s. (p.v.v.s.) of a semigroup, or of semigroups, we will write v.v.s.s. (p.v.v.s.s.).

The following two propositions are clear.

1.1. *The class of $(n, 1)$-subgroupoids of semigroups coincides with the class of proper $(n, 1)$-subgroupoids of semigroups.* □

1.2. *An (n, n)-groupoid* (Q; f) *is a v. v. s. s. iff $f(x_1, \dots, x_n) = (x_1, \dots, x_n)$, i.e. f is the identity transformation of Q^n.* □

Below we give general characterisations of v.v.s.s., and of p.v.v.s.s. as well. We begin by a construction of an enveloping semigroup for a vector valued groupoid.

If A is a non empty set then we denote by A^+ the free semigroup on A, i.e. $A^+ = \{a_1 a_2 \dots a_p \mid a_\nu \in A, p \geqslant 0\}$. If we adjoin the empty sequence 1 to A^+, then we get the free monoid A^* on A. Below we will write a_α^β instead of $a_\alpha a_{\alpha+1} \dots a_\beta$ if $\alpha \leq \beta$, and $\alpha_\alpha^\beta = 1$ if $\beta < \alpha$. We can also assume that $A^p \subseteq A^+$, for every $p \geqslant 1$; namely, we put $A^p = \{a_1^p \mid a_\nu \in A\}$.

Let $(Q;f)$ be an (n, m)-groupoid. (Now we write $f(a_1^n) = b_1^m$ instead of $f(a_1, \dots, a_n) = (b_1, \dots, b_m)$.) We define a relation $\vdash$ on Q^+ in the following way:

$$u \vdash v \Leftrightarrow (\exists x, y \in Q^*, a_\nu, b_\lambda \in Q) \;\; u = x a_1^n y, \;\; v = x b_1^m y, \;\; f(a_1^n) = b_1^m. \tag{1.1}$$

$\sim$ denotes the symmetric closure of $\vdash$, and $\approx$ the reflexive and transitive closure of $\sim$. Thus, if $u, v \in Q^+$, then:

$$\begin{gathered} u \sim v \Leftrightarrow u \vdash v \text{ or } v \vdash u \\ u \approx v \Leftrightarrow (\exists u_0, \dots, u_p \in Q^+, p \geqslant 0), \\ u = u_0, \; v = u_p, \; u_{i-1} \sim u_i, \; i \in \{1, \dots, p\}. \end{gathered} \tag{1.2}$$

The following proposition is clear.

1.3. *The relation $\approx$ is a congruence on Q^+, and the corresponding factor semigroup $Q^\wedge = Q^+/\approx$ is given by the following presentation*

$$< Q; \{a_1, \dots a_n = b_1 \dots b_m \mid f(a_1^n) = b_1^m \text{ in } (Q;f)\} > \tag{1.3}$$

in the class of all semigroups.(We say that $Q^\wedge$ is the universal enveloping semigroup for $(Q;f)$.) □

The class of v.v.s.s. can be characterized by mean of the relation $\approx$ in the following way:

1.4. $(Q;f)$ *is a v.v.s.s. iff*

$$(\forall a, b \in Q)\ (a \approx b \Rightarrow a = b) \tag{1.4}$$

Proof: Let $(Q;f)$ be such that (1.4) holds. Then we can assume that $Q \subseteq Q^\wedge$. Moreover, if $f(a_1^n) = b_1^m$, then we have $a_1^n \approx b_1^m$ in Q^+, and therefore $a_1 \ldots a_n = b_1 \ldots b_m$ in $Q^\wedge$. Thus, $(Q;f)$ is a v.v.s. of $Q^\wedge$.

Conversely, let $(Q;f)$ be a v.v.s. of a semigroup $(S;\bullet)$. Then each defining relation of the presentation (1.2) holds in $(S;\bullet)$, and this implies that there is a homomorphism $\phi : Q^\wedge \rightarrow S$, which is an extension of the inclusion mapping $a \rightarrow a$ from Q into S. If $a, b \in Q$ are such that $a \approx b$, then $a = \phi(a) = \phi(b) = b$, and this shows that (1.4) holds. □

1.5. *Let* $(Q;f)$ *be such that (1.4) is satisfied.* $(Q;f)$ *is a p.v.v.s.s. iff the following condition holds.*

$$(\forall a_\nu, b_\lambda \in Q)\ (a_1^n \approx b_1^m \Rightarrow f(a_1^n) = b_1^m) \tag{1.5}$$

Proof: Let (1.4) and (1.5) be satisfied. By **1.4**, $(Q;f)$ is a v.v.s. of $Q^\wedge$. If $a_\nu, b_\lambda \in Q$ are such that $a_1 \ldots a_n = b_1 \ldots b_m$ in $Q^\wedge$, then we have $a_1^n \approx b_1^m$, and by (1.5) this implies $f(a_1^n) = b_1^m$. This shows that $(Q;f)$ is a p.v.v.s. of $Q^\wedge$.

Assume now that $(Q;f)$ is a p.v.v.s. of a semigroup $(S;\bullet)$. Let $\phi: Q^\wedge \rightarrow S$ be the homomorphism defined as in the second part of the proof of **1.4**. If $a_1^n \approx b_1^m$ in Q^+, then we have $a_1 \ldots a_n = b_1 \ldots b_m$ in $Q^\wedge$, and this implies $a_1 \bullet \ldots \bullet a_n = b_1 \bullet \ldots \bullet b_m$ in $(S;\bullet)$. From the last equation and the fact that $(Q;f)$ is a p.v.v.s. of $(S;\bullet)$ it follows $f(a_1^n) = b_1^m$. □

From the proofs of the last two properties it follows that the enveloping semigroup of a v.v. groupoid has the following universal property.

1.6. *A v.v. groupoid* $(Q;f)$ *is a v.v.s.s. (p.v.v.s.s.) iff* $(Q;f)$ *is a v.v.s. (p.v.v.s.) of* $Q^\wedge$. *If* $(Q;f)$ *is a v.v.s. (p.v.v.s.) of a semigroup* $(S;\cdot)$ *then the inclusion mapping of* Q *in* $Q^\wedge$ *can be extended to a homomorphism* $\phi: Q^\wedge \rightarrow S$. □

As a corollary to **1.4.** we obtain the following proposition:

1.7. *If* $n, m \geqslant 2$, *then every* (n, m)*-groupoid is a v.v.s.s.*

Proof: Let $(Q;f)$ be an (n,m)-groupoid, and let $n, m \geqslant 2$. If $a \in Q$, then there does not exist a $u \in Q^+$ such that $a \sim u$, and this implies that (1.4) is satisfied. Therefore $(Q;f)$ is a v.v.s. of $Q^\wedge$. □

If $m, k \geqslant 1$, then an $(m+k, m)$-groupoid $(Q;f)$ is said to be an $(m+k, m)$-s e m i g r o u p iff the following equation

$$f(f(x_1^{m+k})x_{m+k+1}^{m+2k}) = f(x_1^i f(x_{i+1}^{i+m+k})\ x_{i+m+k+1}^{m+2k}), \tag{1.6}$$

is an identity for any $i \in \{1, \ldots, k\}$. □

1.8. *An* $(m+k, m)$*-groupoid* $(Q;f)$ *is a p.v.v.s.s. iff it is an* $(m+k, m)$*-semigroup.*

Proof: It is well known (see for example [3]), that an $(n,1)$-groupoid is an $(n,1)$-subgroupoid of a semigroup iff it is an $(n, 1)$-semigroup, i.e. an n-semigroup. In the paper [2] it is proved that an $(m+k, m)$-semigroup is a proper $(m+k, m)$-subgroupoid of $Q^\wedge$.

Assume that an $(m+k, m)$-groupoid $(Q;f)$ is not an $(m+k, m)$-semigroup. Then, there exists an $e_1^{m+2k} \in Q^{m+2k}$ and an $i \in \{1, \ldots, k\}$ such that:

$$f(f(e_1^{m+k})\ e_{m+k+1}^{m+2k}) \neq f(e_1^i f(e_{i+1}^{i+m+k})\ e_{i+m+k+1}^{m+2k}).$$

Let

$$f(e_1^{m+k}) = a_1^m,\ e_{m+k+1}^{m+2k} = a_{m+1}^{m+k}, \qquad f(e_1^i f(e_{i+1}^{i+m+k})\ e_{i+m+k+1}^{m+2k}) = b_1^m.$$

Then we have $a_1^{m+k} \approx b_1^m$, but $f(a_1^{m+k}) \neq b_1^m$, and thus (1.5) is not satisfied. □

2. Superpositions and direct products of vector valued operations have the usual meanings[1]. If Q is a set and r a positive integer then we denote by 1^r the identity mapping on Q^r.

Let $(Q;f)$ be an (n, m)-groupoid, and let $s, t, i_1, \ldots, i_s, j_1, \ldots, j_s$ be non-negative integers such that

$$n+t+s(m-n) \geqslant 1, \quad i_{\nu+1}+j_{\nu+1} = t+\nu(m-n) \tag{2.1}$$

for every $\nu \in (0,1, \ldots, s-1)$. Then the following vector valued operation on Q is well defined:

$$(1^{i_s} \times f \times 1^{j_s}) \ldots (1^{i_2} \times f \times 1^{j_2}) \; (1^{i_1} \times f \times 1^{j_1}) \tag{2.2}$$

Namely, it is an $(n+t, n+t+s(m-n))$-operation on Q. In the case $s = 0$, this operation is 1^{n+t}. We denote the set of all the operations of the form (2.2) by $\mathcal{P}(t, f, s)$, or simply by $\mathcal{P}(t, s)$; we will also write $\mathcal{P}(s)$ instead of $\mathcal{P}(0, s)$.

If $t \geqslant 0$, $s \geqslant 1$ are such that there does not exist $i_1, \ldots, i_s, j_1, \ldots, j_s$ which satisfy (2.1) then $\mathcal{P}(t, s)$ will be the empty set.

The following proposition is clear.

2.1. *$\mathcal{P}(t, s)$ is a finite set for any $t, s \geqslant 0$. If $n \leqslant m$, then $\mathcal{P}(t, s) \neq \emptyset$ for any $t, s \geqslant 0$, and if $m < n$, then $\mathcal{P}(t, s) \neq \emptyset$ iff $n+t > s\,(n-m)$.* □

From the definition of the relation $\vdash$, given in **1**, it follows that

$$u \vdash v \Leftrightarrow (\exists i, j \geqslant 0) \quad (1^i \times f \times 1^j)\,(u) = v \tag{2.3}$$

This implies the following proposition:

2.2. *Let $u, v \in Q$. There exists a sequence $u_0, u_1, \ldots, u_s \in Q^+$, such that $u = u_0$, $v = u_s$, $s \geqslant 1$ and $u_{i-1} \vdash u_i$ for any $i \in \{1, \ldots, s\}$ iff there exist a $t \geqslant 0$ and a $g \in \mathcal{P}(t, s)$ such that $g(u) = v$.* □

To the end of this part of the paper it will be assumed that $n = 1$, $m = 1+k \geqslant 2$, i.e. that $(Q;f)$ is an $(1, 1+k)$-groupoid.

2.3. *If $u, v, w \in Q^+$ are such that $u \neq w$, $v \vdash u$, $v \vdash w$, then there exists a $v' \in Q^+$ such that $u \vdash v'$, $w \vdash v'$*[2].

Proof: Let $v = a_1^p$, $a_\nu \in Q$. Then, there exist i, j such that

$$u = a_1^{i-1}\, b_1^m\, a_{i+1}^p, \qquad f(a_i) = b_1^m$$
$$w = a_1^{j-1}\, c_1^m\, a_{j+1}^p, \qquad f(a_j) = c_1^m .$$

Then $i \neq j$, since $w \neq u$, and so we can assume that $i < j$. If we put

$$v' = a_1^{i-1} b_1^m\, a_{i+1}^{j-1}\; c_1^m\, a_{j+1}^p ,$$

then we will obtain that the relations $u \vdash v'$, $w \vdash v'$ are satisfied. □

2.4. *Let $a \in Q$, $b_0^\alpha \in Q^{\alpha+1}$. Then $a \approx b_0^\alpha$ iff there exist $r, s \geqslant 0$, $g \in \mathcal{P}(r+s)$, $h \in \mathcal{P}(\alpha, s)$ such that:* $\alpha = rk, \quad g(a) = h(b_0^\alpha)$.

Proof: Let $a \in Q$, $b_0^\alpha \in Q^{\alpha+1}$ be such that $g(a) = h(b_0^\alpha)$, for some $g \in \mathcal{P}(r+s)$, $h \in \mathcal{P}(p, s)$. Then by **2.2** we have $a \approx b_0^\alpha$.

Assume now that $a \approx b_0^\alpha$. Then there exist $u_0, u_1, \ldots, u_p$, $p \geqslant 0$ such that $a = u_0$, $b_0^\alpha = u_p$ and $u_{i-1} \sim u_i$ for any $i \in \{1, 2, \ldots, p\}$. If $p = 0$, then we have $a = b_0^\alpha$, i.e. $a = b$, and therefore $1(a) = 1(b_0)$. Let $a \neq b_0^\alpha$. Then $p > 0$. Then $u_1 = f(a)$, i.e. $a \vdash u_1$. By a finite number of applications of **2.3** we can obtain that

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[1] Namely, if: $f_\nu : Q^{n_\nu} \to Q^{m_\nu}$, $n = n_1+n_2+\ldots+n_p$, $m = m_1+m_2+\ldots+m_p$, then $f = f_1 \times \ldots \times f_p$: $Q^n \to Q^m$ is defined by:

$$f(\mathbf{a}_1\mathbf{a}_2 \ldots \mathbf{a}_p) = f_1(\mathbf{a}_1)\, f_2(\mathbf{a}_2) \ldots f_p(\mathbf{a}_p)$$

where $\mathbf{a}_\nu \in Q^{n_\nu}$. If $g: Q^\alpha \to Q^\beta$, $h: Q^\beta \to Q^\gamma$, then $hg: Q^\alpha \to Q^\gamma$ is defined by $(hg)(a_1^\alpha) = h(g(a_1^\alpha))$.

[2] D. Dimovski (personal communication)

$$a \vdash u_1 \vdash u_2 \vdash \ldots \vdash u_p = b_0^\alpha ,$$

or

$$a \vdash u_1 \vdash u_2 \vdash \ldots \vdash u_{r+s} \dashv u_{r+s+1} \dashv \ldots \dashv u_{p-1} \dashv b_0^\alpha ,$$

where $p = r + 2s$. By **2.2** there exist $g \in \mathcal{P}(r+s)$, $h \in \mathcal{P}(\alpha, s)$ such that

$$g(a) = u_{r+s} = h(b_0^\alpha) .$$

Then we also have

$$u_{r+s} = g(a) \in Q^{1+(r+s)k}, \qquad u_{r+s} = h(b_0^\alpha) \in Q^{1+\alpha+sk},$$

which implies $\alpha = rk$. □

Now we can give satisfactory descriptions of $(1, 1+k)$-groupoids which are v.v.s.s. (p.v.v.s.s.).

2.5. *An $(1,1+k)$-groupoid $(Q;f)$ is a v.v.s.s. iff for every $s \geqslant 0$ and $g, h \in \mathcal{P}(s)$, the following quasi-identity*

$$g(x) = h(y) \Rightarrow x = y \tag{2.4}$$

is satisfied.

Proof: Assume that $(Q;f)$ satisfies every quasi-identity (2.4). Let $a, b \in Q$ be such that $a \approx b$. By **2.4** there exist an $s \geqslant 0$ and g, $h \in \mathcal{P}(s)$ such that $g(a) = h(b)$. Then by (2.4) we have $a = b$. Thus $(Q;f)$ satisfies (1.4), and by **1.4** $(Q;f)$ is a v.v.s. of $Q^\wedge$.

Let $(Q;f)$ be a v.v.s.s., and let $a, b \in Q$, $s \geqslant 0$, $g, h \in \mathcal{P}(s)$ be such that $g(a) = h(b)$. Then by **2.2** we have $a \approx b$, and this by (1.4) implies $a = b$. Thus, every quasi-identity (2.4) is satisfied. □

2.6. *Let $(Q;f)$ be an $(1,1+k)$-subgroupoid of a semigroup. Then $(Q;f)$ is a p.v.v.s.s. iff for any $s \geqslant 0$, $g \in \mathcal{P}(s+1)$, $h \in \mathcal{P}(k, s)$ the following quasi-identity*

$$g(x) = h(y_0^k) \Rightarrow f(x) = y_0^k \tag{2.5}$$

is satisfied.

Proof: We have to make the same discussion as in the proof of **2.5**. Namely, this proposition is a corollary of **1.5**, **2.2** and **2.4**. □

3. Here we make some remarks and state some problems.

3.1) We have given a complete description of v.v.s.s., but concerning p.v.v.s.s., we do not have a satisfactory description of proper (n, m)-subgroupoids of semigroups in the case $n \geqslant 2$, $m - n = k \geqslant 1$.

Thus if $(Q;f)$ is an $(n, n+k)$-groupoid, where $n \geqslant 2$, $k \geqslant 1$, and if $(Q;f)$ is a p.v.v.s.s. it can easily be seen that for any $s \geqslant 0$ and any $g, h \in \mathcal{P}(s+1)$, $l \in \mathcal{P}(k, s)$ the following quasi-identities

$$g(x_1^n) = h(y_1^n) \Rightarrow x_1^n = y_1^n$$

$$g(x_1^n) = l(y_1^{n+k}) \Rightarrow f(x_1^n) = y_1^{n+k}$$

are satisfied, but we do not know whether these quasi-identities are sufficient for $(Q;f)$ to be a p.v.v.s.s..

3.2) Let n, m be given positive integers and let B be a non-empty set. It is easy to give a description of the free (n, m)-groupoid with a basis B. Namely, let $F_{n,m}(B) = \bigcup_{p \geq 0} B_p$, where $B_0 = B$ and $B_{p+1} = B_p \cup \{(u_1^n, i) \mid u_\nu \in B_p,\ 1 \leqslant i \leqslant m\}$. Then, if we put

$$f(u_1^n) = v_1^m \Leftrightarrow v_i = (u_1^n, i),\ i \in \{1, 2, \ldots, m\}$$

we get an (n, m)-groupoid $(F_{n,m}(B);f)$ which is freely generated by B. If $n, m \geqslant 2$, it is a v.v.s.s., but if $n - m = k \geqslant 1$, then it is not a p.v.v.s.s., for it is not an $(m+k, m)$-semigroup. We do not know whether $(F_{n,m}(B);f)$ is a p.v.v.s.s. in the case $n < m$.

3.3) It is natural to ask for "Mal'cev-like" axiom systems of the class of v.v.s. (p.v.v.s.) of groups ([1], [5]).

3.4) Let F be a set of vector valued operations on a set A. Then $(A;F)$ is called a vector valued algebra (v.v.a.). We say that $(A;F)$ is an F-subgroupoid of a semigroup $(S;\cdot)$ iff, for any $f \in F$, $(A;f)$ is a v.v.s. of $(S;\cdot)$. It is not difficult to see that the corresponding generalizations of 1.1, 1.2, 1.3, 1.4 and 1.5 hold. (We note that if $(A;F)$ is a v.v.a., then the relation $\vdash$ on A^+ is defined by

$$u \vdash v \Leftrightarrow (\exists x, y \in A^*, f \in F)\ u = xa_1^n y,\ \ v = xb_1^m y,\ f(a_1^n) = b_1^m \qquad (1.1')$$

and $\approx$ is the reflexive, symmetrical and transitive closure of $\vdash$ on A^+. Then $\approx$ is a congruence on A^+ and the factor semigroup $A^+/\approx$ is given by the following presentation:

$$< A; \{a_1 \ldots a_n = b_1 \ldots\ b_m \mid f(a_1^n) = b_1^m \text{ in } (A;F)\} > \qquad (1.3')$$

It is known that in the general case the corresponding generalization of **1.8** is not true. That is why we find it interesting to look for an axiom system of the class of v.v. F-subgroupoid of semigroups.

3.5) If $n, m \geqslant 1$, $s, t \geqslant 0$ are such that $n+t+s(m-n) \geqslant 1$, then it can easily be seen that

$$|\mathcal{P}(t, s)| \leqslant \prod_{\nu=0}^{s-1} [1+t+\nu(m-n)].$$

Assume that λ and μ are two integers such that:

$$\mu \geqslant 2,\ 1 \leqslant \lambda \leqslant \prod_{\nu=0}^{s-1} [1+t+\nu(m-n)].$$

Does there exist an (n, m)-groupoid $(Q;f)$ such that $|Q| = \mu$, $|\mathcal{P}(t, s)| = \lambda$?

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ВЕКТОРСКО ВРЕДНОСНИ ПОДГРУПОИДИ ОД ПОЛУГРУПИ

(Резиме)

Во работава се разгледуваат две класи векторско вредносни групоиди, а тоа се, имено, класата векторско вредносни подгрупоиди од полугрупи, како и нејзината поткласа — класата чисти векторско вредносни подгрупоиди од полугрупи. Се дава комплетен опис на првата класа, како и скоро комплетен опис на втората класа. На крајот на работава се формулираат неколку нерешени проблеми.