MULTIDIMENSIONAL ASSOCIATIVES

Мат. билтен Македонија, 13 (39) (1989), 5–14

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Abstract. Let M be a set of positive integers. For every $m\in M$, let F_m be a set of vector valued operations on a set A, such that

$$(\forall f\in F_m)\ f:A^{m+k_f} \to A^m,$$ [1)]

where $k_f > 0$. Denote by F the set $\bigcup\{F_m \mid m\in M\}$.

The vector valued algebra (A;F) is said to be an associative if the general associative law holds. In two previous papers ([1] and [2]) some results of associatives concerning the case $|M|=1$ are obtained, and here we make corresponding investigations assuming that M is an arbitrary nonempty set of positive integers.

§1. Polynomial operations

Let A be a nonempty set, and let Op(A) be the set of vector valued operations on A, i.e.

$$\mathrm{Op}(A) = \{f:A^n \to A^m \mid n,m \geq 1\}.$$

If $f:A^n \to A^m$, then we write $\delta f=n$, $\rho f=m$, $\iota f=n-m$ (or $\delta(f)=n$ etc. when parenthesis are more convenient), and we say that n,m, n-m are the length, dimension, index of f - respectively.

Let F be a nonempty subset of Op(A) with the following property:

$$(\forall f\in F)\ (\iota f=\delta f-\rho f > 0). \tag{1.1}$$

We define a set of operations $\mathcal{P}(F)\subseteq \mathrm{Op}(A)$, which will be called the set of polynomials generated by F, in the following way:

$$\mathcal{P}(F) = \bigcup\{F_\alpha \mid \alpha \geq 1\}, \tag{1.2}$$

where:

$$F_1 = F,$$

$$F_{\alpha+1} = F_\alpha \cup \{g(g_1\times\dots\times g_p) \mid g\in F_\alpha, g_\nu\in F_\alpha\cup\{1_A\}, \delta g = \sum_{\nu=1}^{p}\rho g_\nu\}$$ [2)]

It can be easily shown that:

P.1.1. $\mathcal{P}(\mathcal{P}(F)) = \mathcal{P}(F)$. ♦

It is desirable to have a corresponding description of the sets $\rho(\mathcal{P}(F))$, $\iota(\mathcal{P}(F))$[3)].

First we have:

P.1.2. $\rho(\mathcal{P}(F)) = \rho(F)$. ♦

Denote the set $\rho(F)$ by:

$$M = \{m_1, m_2, \dots\} = \{m_\lambda \mid \lambda\in\Lambda\}, \tag{1.3}$$

where $m_\nu < m_{\nu+1}$, and Λ is the set of positive integers or $\Lambda=\{1,2,\dots,t\}$. We assume that $|\Lambda| \geq 2$, for the case $\Lambda=\{1\}$ is considered in [2] and [4].

Denote by F_λ ($\mathcal{P}_\lambda(F)$) the set of elements $f\in F$ ($f\in\mathcal{P}(F)$) such that $\rho f=m_\lambda$ and put:

$$\iota(F_\lambda)=I_\lambda,\ \iota(F)=I,\ \iota(\mathcal{P}_\lambda(F))=K_\lambda,\ \iota(\mathcal{P}(F))=K \tag{1.4}$$

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1) A^r is the r-th Cartesian power of A.

2) Composition and direct products of operations have the usual meanings; $1_A=1$ is the identity transformation of A (see for ex. [4;123-124]).

3) If G is a set of operations on A, and τ is a mapping from G into a set B, then $\tau(G)=\{\tau(g) \mid g\in G\}$.

Clearly:

P.1.3. $I=\bigcup\{I_\lambda \mid \lambda\in\Lambda\}$, $K=\bigcup\{K_\lambda \mid \lambda\in\Lambda\}$. ♦

By a usual induction it can be shown that:

P.1.4. K is an additive semigroup of positive integers generated by I, and, for every $\lambda\in\Lambda$, K_λ is a subsemigroup of K. ♦

Assume now that $\mu,\nu\in\Lambda$ and $k_\nu\in K_\nu$ are such that $m_\nu+k_\nu \geq m_\mu$, and let k_μ be an arbitrary element of K_μ. Then, there exist $f\in\mathcal{P}_\nu(F)$, $g\in\mathcal{P}_\mu(F)$ such that $f=k_\nu$, $g=k_\mu$ and

$$h = f(g\times\underbrace{1\times\ldots\times 1}_{m_\nu+k_\nu-m_\mu})\in\mathcal{P}_\nu(F), \quad h=k_\nu+k_\mu.$$

This implies the following property of the collection $\{K_\lambda \mid \lambda\in\Lambda\}$:

P.1.5. If $\nu,\mu\in\Lambda$, $k_\nu\in K_\nu$ are such that $k_\nu+m_\nu\geq m_\mu$, then $k_\nu+K_\mu\subseteq K_\nu$. ♦

Now we will give a satisfactory description of the collection of semigroups $\{K_\lambda \mid \lambda\in\Lambda\}$.

P.1.6. Let a collection of sets of positive integers $\{I_{\nu,\alpha} \mid \lambda\in\Lambda,\ \alpha\geq 0\}$ be defined as follows:

$$I_{\nu,o} = I_\nu, \quad I_{\nu,\alpha+1} = I_{\nu,\alpha}\cup \bar{I}_{\nu,\alpha},$$

where

$$\bar{I}_{\nu,\alpha} = \{i_\nu+i_\lambda \mid i_\nu\in I_{\nu,\alpha},\ i_\lambda\in I_\lambda,\ m_\nu+i_\lambda\geq m_\lambda,\ \lambda\in\Lambda\}.$$

Then:

$$K_\nu = \bigcup\{I_{\nu,\alpha} \mid \alpha\geq 0\}. \quad ♦$$

If $F\subseteq Op(A)$, then F induces a vector valued algebra $\mathcal{A}=(A;F)$ with a carrier A. Then $\mathcal{P}(\mathcal{A})=(A;\mathcal{P}(F))$ is the corresponding polynomial algebra. It is clear that:

P.1.7. If $C\subseteq A$, $C\neq\emptyset$, then:

C is a subalgebra of $\mathcal{A}$ iff C is a subalgebra of $\mathcal{P}(\mathcal{A})$. ♦

Let A' be a set and $F'\subseteq Op(A')$. A homomorphism from $\mathcal{A}$ into $\mathcal{A}'=(A';F')$ is a pair of mappings $\zeta:A\to A'$, $\psi:F\to F'$ such that ψ is surjective and:

$$(\forall f\in F)(\delta f=\delta(\psi(f)),\ \rho f=\rho(\psi(f))),$$

$$(\forall a_\nu\in A,\ f\in F)\ (\zeta(f(a_1^n))=f'(\bar{a}_1^n)),$$

where $\zeta(c)=\bar{c}$, $\psi(f)=f'$, and $\zeta(b_1^m)=\bar{b}_1^m$.

P.1.8. Every homomorphism (ζ,ψ) from $(A;F)$ into $(A';F')$ induces a unique homomorphism (ζ,ψ) from $(A;\mathcal{P}(F))$ into $(A';\mathcal{P}(F'))$. ♦

§2. Associatives

As in the previous section, we will assume that $A\neq\emptyset$ and $F\subseteq Op(A)$, $F\neq\emptyset$, is such that $\ell f>0$, i.e. $\delta f>\rho f$, for every $f\in F$.

We say that F is an associative on A iff the following condition is satisfied:

$$f,g\in\mathcal{P}(F),\ \delta f=\delta g,\ \rho f=\rho g \Longrightarrow f=g. \tag{2.1}$$

Let K, $\{K_\lambda \mid \lambda\in\Lambda\}$ and $M=\{m_\lambda \mid \lambda\in\Lambda\}$ be defined as in the previous section. By P.1.1 one obtains:

P.2.1. F is an associative on A iff $\mathcal{P}(F)$ is an associative on A. ♦

According to this proposition, we will assume further on that

$$\mathcal{P}(F) = F. \tag{2.2}$$

Therefore, for every $k_\lambda\in K_\lambda$, $\lambda\in\Lambda\setminus\{0\}$, there exists a unique $f^{(k_\lambda,m_\lambda)}:A^{k_\lambda+m_\lambda}\to A^{m_\lambda}$ (where $K_o=\{0\}$, $m_o=1$, $f^{(o,1)}=1_A$). This enables us, for every $\lambda\in\Lambda$, to define a unique mapping

$$f^{(m_\lambda)}: A^{K_\lambda + m_\lambda} \to A^{m_\lambda}\text{ [4]}$$

by

$$(\forall x \in A^{k_\lambda + m_\lambda}) f^{(m_\lambda)}(x) = f^{(k_\lambda, m_\lambda)}(x).$$

Thus one obtains a set of mappings

$$G = \{f^{(m_\lambda)}: A^{K_\lambda + m_\lambda} \to A^{m_\lambda} \mid \lambda \in \Lambda\} \tag{2.3}$$

with the following property:

$$m_{\lambda_1} + \ldots + m_{\lambda_p} \in K_\mu + m_\mu \Longrightarrow f^{(m_\mu)}(f^{(m_{\mu_1})} \times \ldots \times f^{(m_{\mu_p})}) \subseteq f^{(m_\mu)}\text{ [5]} \tag{2.4}$$

And conversely:

If a family of mappings (2.3) has the property (2.4) and

$$f^{(k_\lambda, m_\lambda)} = f^{(m_\lambda)} \Big|_{A^{k_\lambda + m_\lambda}},$$

then the set

$$F = \{f^{(k_\lambda, m_\lambda)} \mid \lambda \in \Lambda,\ k_\lambda \in K_\lambda\} \tag{2.5}$$

is an associative on A.

Note that, by induction, it is easy to show that the condition (2.4) can be changed with the following special (weaker) condition:

$$\alpha + m_\lambda + \beta \in K_\mu + m_\mu \Longrightarrow f^{(m_\mu)}(1^\alpha \times f^{(m_\mu)} \times 1^\beta) \subseteq f^{(m_\mu)}. \tag{2.6}$$

Further on we will always consider the class of F-associatives as a class of mappings (2.3), which satisfy (2.4), where K, $\{K_\lambda \mid \lambda \in \Lambda\}$, M have the above mentioned properties. Instead of "F-associative", we will write "$(K;\{K_\lambda \mid \lambda \in \Lambda\};M)$-associative", and we will say that $\Phi = (K;\{K_\lambda \mid \lambda \in \Lambda\};M)$ is the <u>type</u> of the associative. Also we will write

$$\left[a_1^{k_\lambda + m_\lambda}\right]^{(\lambda)} \text{ instead of } f^{(m_\lambda)}(a_1^{k_\lambda + m_\lambda}).$$

Assume that A and A′ are the carriers of two associatives of the same type Φ. A mapping $\zeta: c \mapsto \bar{c}$ from A into A′ is a homomorphism iff:

$$\left[a_1^{k_\lambda + m_\lambda}\right]^{(\lambda)} = b_1^{m_\lambda} \Longrightarrow \left[\bar{a}_1^{k_\lambda + m_\lambda}\right]^{(\lambda)} = \bar{b}_1^{m_\lambda}. \tag{2.7}$$

It can be easily seen that this definition of homomorphism is compatible with the usual definition given in §1.

§3. Free associatives

The notion of a "free associative with a basis B" has the usual meaning. So we will not state here the corresponding explicit definition, but we will give a construction of free associatives.

Let B be a nonempty set and $\Phi = (K;\{K_\lambda \mid \lambda \in \Lambda\};M)$ a type of associatives, $M = \{m_1, m_2, \ldots\}$, $m_\lambda < m_{\lambda+1}$. Denote

$m_1 + \ldots + m_\lambda$ by $\bar{m}_\lambda$, $\{1,2,\ldots,t\}$ by N_t, $\bar{m}_0 = 0$, $N_0 = \emptyset$.

Define a sequence of sets $(B_\alpha \mid \alpha \geq 0)$ as follows: $B_0 = B$ and

$$B_{\alpha+1} = B_\alpha \cup C_\alpha, \tag{3.1}$$

where

$$C_\alpha = \bigcup \{(N_{\bar{m}_\lambda} \setminus N_{\bar{m}_{\lambda-1}}) \times R_{\alpha,\lambda} \mid \lambda \in \Lambda\}. \tag{3.2}$$

Now we have to explain the meaning of $R_{\alpha,\lambda}$. First, we define $R_{\alpha,\lambda}$ by:

$$R_{0,\lambda} = (B_0)^{K_\lambda + m_\lambda}. \tag{3.3}$$

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[4] We note that if P is a set of positive integers on a set A, then $A^P = \cup\{A^p \mid p \in P\}$. (Thus, here, A^P has not the usual meaning - the set of all mappings from P into A.)

[5] If $f: B \to D$, $g: C \to D$, then $f \subseteq g$ iff $B \subseteq C$ and $(\forall x \in B)$ $f(x) = g(x)$, i.e. f is the restriction of g on B.

Assume that B_α is well defined. Then (as usually) we denote by B_α^+ the free semigroup with a basis B and by $B_\alpha^* = B_\alpha^+ \cup \{1\}$ the free monoid with a basis B. An element

$$u = (\overline{m}_\lambda+1,y)(\overline{m}_\lambda+2,y)\dots(\overline{m}_{\lambda+1},y)\in B_\alpha^+ \qquad (3.4)$$

is called an elementary reduction, and an element $c\in B_\alpha^+$ is said to be reducible iff $x = x'ux''$, where $x',x''\in B_\alpha^*$ and u is an elementary reduction. And, $x\in B_\alpha^+$ is said to be reduced if it is not reducible. Then $R_{\alpha,\lambda}$ is the set of all the reduced elements of $B_\alpha^{m_\lambda+K_\lambda}$.

Thus, $R_{\alpha,\lambda}$ is a well defined subset of $B_\alpha^{m_\lambda+K_\lambda}$ for every $\alpha \geq 1$, $\lambda\in\Lambda$. Moreover, if $x\in B_\gamma$, then $x\in R_{\gamma,\lambda}$ iff $x\in R_{\gamma+1,\lambda}$.

Denote the set $\bigcup\{B_\alpha \mid \alpha\geq 0\}$ by $\overline{B}$.

If $x\in\overline{B}^+$, then we say that $x\in\overline{B}$ is reduced if $x\in B_\alpha^+$ and x is reduced in B_α^+. Denote the set of the reduced elements of $\overline{B}^+$ by R. Thus, $R=\bigcup\{R_\alpha \mid \alpha\geq 0\}$, where R_α is the set of reduced elements of B_α^+.

The concepts of hierarchy $\mathcal{H}$ in $\overline{B}$ and norme $|\ |$ in $\overline{B}^+$ are defined as follows:

$$\mathcal{H}(u) = \min\{\alpha \mid u\in B_\alpha\};$$

$$|u| = 0 \iff u\in B^+, \quad |(i,x)| = 1 + |x|, \quad |xy| = |x| + |y|.$$

Now we will define a mapping $\psi:\overline{B}^+\to R$ in the following way ((i) and (ii)):

(i) $x\in R \implies \psi(x) = x$.

Let $x\in\overline{B}^+\setminus R$ and let $\psi(y)\in R$ be defined for every $y\in\overline{B}^+$ such that $|y| < |x|$, and then

$$\psi(y) \neq y \iff |\psi(y)| < |y|. \qquad (3.5)$$

Let $x=x'ux''$, where u is an elementary reduction of the form (3.4), and x' is reduced (or $x'=1$). Then $|x'yx''| < |x|$, and thus $\psi(x'yx'')$ is well defined. Then we define $\psi(x)$ by:

(ii) $\psi(x) = \psi(x'yx'')$.

Then we have:

$$|\psi(x)| = |\psi(x'yx'')| \leq |x'yx''| < |x|,$$

and this implies that $\psi:\overline{B}^+ \to R$ is a well defined mapping such that (3.5) is satisfied.

Let us establish some properties of the mapping ψ.

P.3.1. If $x\in\overline{B}^+$ and $dm(x)\in m_\nu+K_\nu$,[6] then $dm(\psi(x))\in m_\nu+K_\nu$.

Proof. Let $\psi(x)$ be defined by (ii). Then, $dm(x)=dm(x'x'')+m_{\lambda+1}$,

$$dm(x'yx'') = dm(x'x'') + dmy.$$

The fact that $(\overline{m}_\lambda+i,y)\in\overline{B}$ $(1\leq i\leq m_{\lambda+1})$ implies that $dmy=k_{\lambda+1}+m_{\lambda+1}$ for some $k_{\lambda+1}\in K_{\lambda+1}$. But $dmx\in m_\nu+K_\nu$ implies that $dmx=m_\nu+k_\nu$ for some $\nu\in\Lambda$. Thus we have: $m_\nu+k_\nu = dm(x'x'')+m_{\lambda+1}$, and therefore $m_\nu+k_\nu \geq m_{\lambda+1}$, which implies that $k_\nu + K_{\lambda+1} \subseteq K_\nu$. Finally, we obtain

$$dm(x'yx'')=dm(x'x'')+dm(y)=dm(x'x'')+k_{\lambda+1}+m_{\lambda+1}=m_\nu+k_\nu+k_{\lambda+1}\in m_\nu+K_\nu.$$

Then, by an induction on norms, we obtain that

$$dm(\psi(x)) = dm(\psi(x'yx''))\in m_\nu+K_\nu. \quad \blacklozenge$$

P.3.2. $(\forall x\in\overline{B}^+,\ y\in\overline{B}^*)\ \psi(xy) = \psi(\psi(x)y)$. ♦

P.3.3. If $x=x'ux''$, $x',x''\in\overline{B}^*$, and u is an elementary reduction of the form (3.4), then $\psi(x) = \psi(x'yx'')$.

Proof. If $x'=1$ or $x'\in R$, then the above equality holds by (ii), and if $x\neq 1$, $x\notin R$, then we can apply P.3.2 and an induction on norms. ♦

P.3.4. $(\forall x',x''\in\overline{B}^*,\ x\in\overline{B}^+)\ \psi(x'xx'') = \psi(x'\psi(x)x'')$.

Proof. We can assume that $\psi(x)\neq x$, and apply P.3.3. ♦

Now, we will define a collection of mappings

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[6] If $x\in\overline{B}^\alpha\subset\overline{B}^+$, then $dm(x)=\alpha=dmx$.

$$\{[\ ]^{\lambda}:\bar{B}^{m_\lambda+K_\lambda}\to\bar{B}^{m_\lambda}\mid\lambda\in\Lambda\}.$$

Namely, assume that $\lambda\in\Lambda$ and $x\in\bar{B}^{m_\lambda+K_\lambda}$. Then, by P.3.1, $\psi(x)\in\bar{B}^{m_\lambda+K_\lambda}$, and thus $z_i=(\bar{m}_{\lambda-1}+i,\psi(x))\in\bar{B}$, for every $i\in N_{m_\lambda}$. Then we put

$$[x]^{\lambda}=z_1^{m_\lambda}. \tag{3.6}$$

To show that $(\bar{B};\{[\ ]^{\lambda}\mid\lambda\in\Lambda\})$ is a $(K;\{K_\lambda\mid\lambda\in\Lambda\};\{m_\lambda\mid\lambda\in\Lambda\}$-associative, we have to show that

$$[x'[x]^{\lambda}\ x'']^{\lambda}=[x'xx'']^{\lambda}, \tag{3.7}$$

for every pair $\nu,\lambda\in\Lambda$, and $x',y,x''\in\bar{B}^*$ such that the left hand side is well defined.

Namely, first we have that $x\in\bar{B}^{m_\lambda+K_\lambda}$ and

$$[x]^{\nu}=(\bar{m}_{\nu-1}+1,\psi(x))\dots(\bar{m}_{\nu-1}+m_\nu,\psi(x)).$$

By P.3.4 we have $\psi(x'[x]^{\nu}\ x'')=\psi(x'xx'')$, and this implies that (3.7) is satisfied.

Thus, $(\bar{B};\{[\ ]^{\lambda}|\lambda\in\Lambda\})$ is an associative. By induction on hierarchy, it can be easily seen that B is a generating subset of this associative.

Assume that $(Q;\{[\ ]^{\lambda}|\lambda\in\Lambda\})$ is an associative of the same type and $\zeta:B\to Q$ an arbitrary mapping. Define a sequence of mappings $\{\zeta_\alpha:B_\alpha\to Q\}$ as follows.

First, we put $\zeta_0=\zeta$. Assume that $\zeta_\alpha:B_\alpha\to Q$ is well defined. Let $v\in B_{\alpha+1}\setminus B_\alpha$. Then there exist $\lambda+1\in\Lambda$ and $i\in N_{\lambda+1}$ such that $v=(\bar{m}_\lambda+i,x)$, where $x\in R_{\lambda+1,\alpha}$. Thus, $x=u_1u_2\dots u_{m_{\lambda+1}+k_{\lambda+1}}$, for some $u_j\in B_\alpha$. Then $\zeta_\alpha(u_j)\in Q$ is well defined. Let

$$[\![\zeta_\alpha(u_1)\dots(\zeta_\alpha(u_{m_{\lambda+1}+k_{\lambda+1}})]\!]=c_1^{m_{\lambda+1}}.$$

Then we put $\zeta_{\alpha+1}(v)=c_i$, and $\zeta_{\alpha+1}(w)=\zeta_\alpha(w)$, for every $w\in B_\alpha$.

Define by ζ the uniquelly determined mapping $\psi:\bar{B}\to Q$ which is an extension of ζ_α, for every $\alpha\ge 0$.

By induction it can be shown that

$$\psi:(\bar{B};\{[\ ]^{\lambda}|\lambda\in\lambda\})\to(Q;\{[\![\ ]\!]^{\lambda}\mid\lambda\in\Lambda\})$$

is a homomorphism, and that will complete the proof of the following

Theorem 3.5. $(Q;\{[\ ]^{\lambda}\mid\lambda\in\Lambda\})$ is a free $(K;\{K_\lambda\mid\lambda\in\Lambda\};\{m_\lambda\mid\lambda\in\Lambda\})$ associative with a basis B. ||

R E F E R E N C E S

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МНОГУДИМЕНЗИОНАЛНИ АСОЦИЈАТИВИ

Р е з и м е

Нека M е множество позитивни цели броеви и, за секој $m\in M$, нека F_m е множество векторско вредносни операции на едно множество A, така што $(\forall f\in F_m)\ f: A^{m+k_f}\to A^m$,

каде што $k_f>0$. (Притоа, A^r означува r-ти декартов степен на множеството A.) Да го означиме со F множеството $\{F_m|\,m\in M\}$.

Векторско вредносната алгебра (A;F) се вика асоцијатив ако важи општиот асоцијативен закон. Во два поранешни труда ([1] и [2]) се добиени некои резултати за асоцијативи што се однесуваат на случајот $|M|=1$, а овде се вршат соодветни испитувања земајќи M да е произволно непразно множество позитивни цели броеви.