VECTOR VALUED GROUPOIDS INDUCED BY VARIETIES OF SEMIGROUPS

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Abstract. Vector valued groupoids induced by semigroups are considered in [3]. Here we consider vector valued groupoids induced by (nontrivial) varieties of semigroups.

Preliminaries. First we state some definitions and results concerning vector valued groupoids induced by semigroups, considered in [3].

Let $\underline{S}=(S;\cdot)$ be a semigroup, and Q a nonempty subset of S. Define a collection of subsets $(Q_\alpha \mid \alpha \geq 1)$ of S by: $Q_1 = Q$, $Q_{\alpha+1} = \{xy \mid x\in Q_\alpha,\ y\in Q\}$. If n and m are positive integers and $f:Q_n \to Q_m$ a mapping from Q_n into Q_m, then the ordered pair $(Q;f)$ is called an $(S;n,m)$-groupoid. Then, a nonempty subset P of Q is said to be a subgroupoid of $(Q;f)$ if $f(P_n)\subseteq P_m$, and P is called a strong subgroupoid of $(Q;f)$ iff

$$(\forall a_i\in P, b_j\in Q)(f(a_1\cdot\ldots\cdot a_n) = b_1\cdot\ldots\cdot b_m \Longrightarrow b_1,\ldots,b_m\in P)$$

If $(Q;f)$ is an $(S;n,m)$-groupoid and $(Q';f')$ is an $(S';n,m)$-groupoid, then a mapping $\phi:Q \to Q'$ is said to be a homomorphism from $(Q;f)$ into $(Q';f')$ if for every $a_i,b_j\in Q$ the equation $f(a_1\cdot\ldots\cdot a_n) = b_1\cdot\ldots\cdot b_m$ implies $f'(\phi(a_1)*\ldots*\phi(a_n)) = \phi(b_1)*\ldots*\phi(b_m)$, where $\underline{S}'=(S';*)$. If, moreover, ϕ is bijective and ϕ^{-1} is a homomorphism then ϕ is called an isomorphism.

We state now some results, proved in [3].

(i) A nonempty intersection of strong subgroupoids is a strong subgroupoid as well, but a nonempty intersection of subgroupoids is not necesseraly a subgroupoid.

(ii) A bijective homomorphism is not necesseraly an isomorphism.

(iii) A homomorphic image of a subgroupoid is a subgroupoid, but a homomorphic image of a strong subgroupoid is not necessarily a strong subgroupoid.

(iv) A complete nonempty homomorphic inverse image of a strong subgroupoid is a strong subgroupoid, but this is not true, in general, for subgroupoids.

Assume now that $\mathbb{V}$ is a nontrivial variety of semigroups. (By "a nontrivial" we mean that $\mathbb{V}$ contains objects with more than one element.) If Q is a nonempty set then we denote by $\mathbb{V}(Q)$ a free semigroup in $\mathbb{V}$ with a basis Q. Every $(\mathbb{V}(Q);n,m)$-groupoid is called a $(\mathbb{V};n,m)$-groupoid. Here, we will write $\mathbb{V}_p(Q)$ instead of Q_p.

All mentioned "positive" properties for semigroup (n,m)-groupoids are, certainly, true for $(\mathbb{V};n,m)$-groupoids; nevertheless, some properties hold in the class of $(\mathbb{V};n,m)$-groupoids, which do not hold in the general case. Below we state some properties of this kind.

(i′) A nonempty intersection of subgroupoids of a $(\mathbb{V};n,m)$-groupoid $(Q;f)$ is a subgroupoid as well. If P is a subgroupoid of $(Q;f)$ and if P is not a strong one, then the strong subgroupoid generated by P coincides with Q.

(ii′) A bijective homomorphism is an isomorphism. (When we say that $\phi:(Q;f) \to (Q';f')$ is a homomorphism then we assume that

both $(Q;f)$ and $(Q';f')$ are $(\mathbb{V};n,m)$-groupoids.)

The corresponding "negative" properties stated in (iii) and (iv) remains "negative", in general, in the class of $(\mathbb{V};n,m)$-groupoids as well.

It is given (in Pr. 2.6) a description of the set of varieties $\mathbb{V}$ for which every subgroupoid of a $(\mathbb{V};n,m)$-groupoid is a strong subgroupoid too.

In the last part of the paper, some connections between $(\mathbb{W};n,m)$-groupoids and $(\mathbb{V};n,m)$-groupoids are described, where is a nontrivial subvariety of $\mathbb{V}$.

Consider some examples.

Example 1. If $\mathbb{V}$=Sem is the variety of all semigroups then a $(\mathbb{V};n,m)$-groupoid is a usual (n,m)-groupoid ([2]).

Example 2. The class of fully commutative groupoids ([4]) is obtained in the case when $\mathbb{V}$=Comsem is the variety of commutative semigroups.

Example 3. Let $\mathbb{V}$=Sl be the variety of semilattices, i.e. idempotent and commutative semigroups, and let Q be a nonempty set. As it is well known, the semigroup $Sl(Q)$ can be interpreted as the semigroup $F(Q)$ of all finite nonempty subsets of Q, where the operation is the usual (set theoretical) union. Then an $(Sl;n,m)$-groupoid can be considered as a mapping $f:X \to Y=f(X)$ from $\{X\in F(Q) \mid 1 \le |X| \le n\}$ into $\{Y\in F(Q) \mid 1 \le |Y| \le m\}$. ($|A|$ denotes the cardinal number of the set A.)

Example 4. Let $\mathbb{V}$=RB be the variety of rectangular bands, i.e. idempotent semigroups satisfying the law $xyz=xz$. Then, $\mathbb{V}_\alpha(B)=B\times B$, for every $\alpha \ge 2$, where an element $a\in Q$ is identified by the pair (a,a) $(=a\cdot a)$. If $1 \le n,m \le 2$ then an $(RB;n,m)$-groupoid is the same as an (n,m)-groupoid, according to Ex. 1. If $n \ge 3$, $m=2$, then the class of $(RB;n,m)$-groupoids coincides with the class of all (n,m)-groupoids which satisfy all the identities of the form

$$f(xz_1\ldots z_{n-2}y) = f(xu_1\ldots u_{n-2}y).$$

We also note that in the first three examples there are not any distinctions between subgroupoids and strong subgroupoids, but, if $m \ge 3$, Q is the unique strong subgroupoid of an $(RB;n,m)$-groupoid $(Q;f)$.

1. Contents in $\mathbb{V}(Q)$. Further on we assume that $\mathbb{V}$ is a given nontrivial variety of semigroups, and Q is a given nonempty set. We will introduce here a notion of a p-content $c_p(u)$ of an element $u\in\mathbb{V}_p(Q)$.

First, let us make some remarks.

(i) Let $a_1,a_2,\ldots$ be a sequence of different elements of Q, and i_λ, j_ν positive integers. Then

$$a_{i_1}\cdot a_{i_2}\cdot\ldots\cdot a_{i_p} = a_{j_1}\cdot a_{j_2}\cdot\ldots\cdot a_{j_q}$$

is an equality in $\mathbb{V}(Q)$ iff

$$x_{i_1}x_{i_2}\ldots x_{i_p} = x_{j_1}x_{j_2}\ldots x_{j_q}$$

is an identity in $\mathbb{V}$.

(ii) Let $u\in\mathbb{V}_p(Q)$, where $p \ge 1$. We define a family $[u;p]$ of subsets of Q as follows.

$A\in[u;p]$ iff there exist $a_1,a_2,\ldots,a_p\in Q$ such that $u=a_1\cdot a_2\cdot\ldots\cdot a_p$ and $A=\{a_1,a_2,\ldots,a_p\}$. (We note that $\{a_1,\ldots,a_p\}$ has the usual meaning, i.e. $a\in\{a_1,\ldots,a_p\} \iff (\exists j)\ a=a_j$.)

Clearly we have $u \in V_p(Q) \Longrightarrow [u;p] \neq \emptyset \ \& \ 0 < |A| \leq p$ for every $A \in [u;p]$.

(iii) If $u \in V_p(Q)$ then $[u;p]$ is a family of finite subsets of Q, and thus for every $L \in [u;p]$ there is at least one minimal element $M \in [u;p]$.

Suppose that M′ and M″ are two different minimal elements of $[u;p]$, and let

$$u = a_1 \cdot a_2 \cdot \ldots \cdot a_p = b_1 \cdot b_2 \cdot \ldots \cdot b_p,$$

where $M' = \{a_1, \ldots, a_p\}$, $M'' = \{b_1, \ldots, b_p\}$. Assume that $b_j \notin M'$ and that $|M''| \geq 2$. Choose an element $b_r \in M''$, such that $b_r \neq b_j$. Define $c_1, \ldots, c_p \in Q$ by:

$$c_i = \begin{cases} b_i & \text{if } i \neq j \\ b_r & \text{if } i = j \end{cases}$$

Then we have $u = c_1 \cdot c_2 \cdot \ldots \cdot c_p$ and $M = \{c_1, \ldots, c_p\}$ is a proper subset of M″, which is impossible. So, if $M'' \setminus M' \neq \emptyset$ then $|M''| = 1$. We obtain symmetrically that $|M'| = 1$. Therefore, we have $u = a^p = b^p$, where $a, b \in Q$, $a \neq b$; furthermore, $u = c^p$ for every $c \in Q$.

In such a way we proved the following

<u>Proposition 1.1</u>. For every positive integer p and every $u \in V_p(Q)$ the set $[u;p]$ either contains least element M or every one element subset of Q is its minimal element. ∎

The last statement suggests the following definition of a p-contents $c_p(u)$ of an element $u \in V_p(Q)$. First we put $c_p(u) = M$ if M is the least element of $[u;p]$, and $c_p(u) = \emptyset$ iff $|Q| \geq 2$ and all one element subsets of Q are minimal members in $[u;p]$.

2. <u>Subgroupoids</u>. We assume here that $(Q;f)$ is a given $(V;n,m)$-groupoid.

<u>Proposition 2.1</u>. If $\{P_i \mid i \in I\}$ is a family of subgroupoids of $(Q;f)$ and if $P = \bigcap \{P_i \mid i \in I\} \neq \emptyset$, then P is a subgroupoid of $(Q;f)$.

<u>Proof</u>. Let $a_1, \ldots, a_n \in P \subseteq P_i$, and let $f(a_1 \cdot \ldots \cdot a_n) = u \in V_m(Q)$. If $c_m(u) = \emptyset$, then we have $u = a^m$ for every $a \in P$, and thus it remains the case when $c_m(u) \neq \emptyset$. The fact that P_i is a subgroupoid implies that there exist $b_{i_1}, \ldots, b_{i_m} \in P_i$ such that $u = b_{i_1} \cdot \ldots \cdot b_{i_m}$. If $M = c_m(u)$ then we have $u = c_1 \cdot \ldots \cdot c_m$, where $A = \{c_1, \ldots, c_m\} \subseteq \{b_{i_1}, \ldots, b_{i_m}\}$ and therefore $M \subseteq P$. ∎

<u>Corollary 2.2</u>. Every nonempty subset B of Q generates a uniquely determined subgroupoid $\langle B \rangle$ of $(Q;f)$. ∎

Now we are going to give a suitable description of $\langle B \rangle$.

<u>Proposition 2.3</u>. Let B be a nonempty subset of Q and define a sequence $(B_\alpha \mid \alpha \geq 0)$ of subsets of Q as follows:

$$B_0 = B, \quad B_{\alpha+1} = B_\alpha \cup \left(\bigcup \{c_m(f(u)) \mid u \in V_n(B_\alpha)\} \right).$$

Then

$$\langle B \rangle = \bigcup \{B_\alpha \mid \alpha \geq 0\}. \ ∎$$

Consider now some connections between subgroupoids and strong subgroupoids.

<u>Proposition 2.4</u>. Let P be a subgroupoid of Q which is not a strong one. If R is a strong subgroupoid of $(Q;f)$ such that $P \subseteq R \subseteq Q$, then $R = Q$.

<u>Proof</u>. The assumption that P is a subgroupoid but not a

strong subgroupoid implies that there exists a $u \in V_n(Q)$ and $b_1,\dots,b_m$, $c_1,\dots,c_m \in Q$ such that

$$f(u) = b_1 \cdot \dots \cdot b_m = c_1 \cdot \dots \cdot c_m,$$

where $b_j \in P$, $c_k \in R$ and there is some i such that $c_i \in R \setminus P$. Let d be an arbitrary element of Q and define a sequence $d_1,\dots,d_m$ by

$$d_k = \begin{cases} c_k & \text{if } k \neq i \\ d & \text{if } k = i \end{cases}$$

Then we have $f(u) = d_1 \cdot \dots \cdot d_m$, which implies that $d = d_i \in R$, i.e. $Q = R$. ∎

If B is a nonempty subset of Q then we denote by $\langle B \rangle_s$ the strong subgroupoid of $(Q;f)$ generated by B. (The existence of $\langle B \rangle_s$ follows from the fact that a nonempty intersection of strong subgroupoids is a strong subgroupoid as well.)

Corollary 2.5. For every nonempty subset B of Q we have $\langle B \rangle_s = \langle B \rangle$ or $\langle B \rangle_s = Q$. ∎

Now we will describe the set of varieties V of semigroups for which there are not any differences between subgroupoids and strong subgroupoids.

Let us say that V is m-regular iff for every nonempty set Q and any element $u \in V_m(Q)$ $|[u;m]| = 1$. In other words, if $a_i, b_j \in Q$ are such that

$$u = a_1 \cdot \dots \cdot a_m = b_1 \cdot \dots \cdot b_m$$

then $(b_1,\dots,b_m)$ is a permutation of $(a_1,\dots,a_m)$.

Proposition 2.6. The following two conditions are equivalent

(a) V is m-regular.

(b) For every $(V;n,m)$-groupoid $(Q;f)$ each subgroupoid of $(Q;f)$ is a strong subgroupoid of $(Q;f)$ as well.

Proof. Let V be m-regular and let P be a subgroupoid of a $(V;n,m)$-groupoid $(Q;f)$. Let $u \in V_n(P)$ and $f(u) = a_1 \cdot a_2 \cdot \dots \cdot a_m$, where $a_i \in Q$. The fact that P is a subgroupoid of $(Q;f)$ implies that $f(u) = b_1 \cdot \dots \cdot b_m$, for some $b_j \in P$. Thus, we have $a_1 \cdot \dots \cdot a_m = b_1 \cdot \dots \cdot b_m$, and from m-regularity of V we obtain that $a_1,\dots,a_m \in P$. Hence, P is a strong subgroupoid of $(Q;f)$.

Assume now that V is not m-regular. Let Q be a set with at least m elements. The assumption that V is not m-regular implies that there exist $a_i, b_j \in Q$ such that $A = \{a_1,\dots,a_m\} \subsetneq \{b_1,\dots,b_m\}$ and $a_1 \cdot \dots \cdot a_m = b_1 \cdot \dots \cdot b_m$ in $V(Q)$.

Define a $(V;n,m)$-groupoid $(Q;f)$ by $f(u) = a_1 \cdot \dots \cdot a_m$ for every $u \in V_n(Q)$. Then A is a subgroupoid of $(Q;f)$, but it is not a strong one. ∎

Certainly, Pr. 2.6 does not mean that if V is not m-regular then the set of strong subgroupoids of a $(V;n,m)$-groupoid $(Q;f)$ is a proper subset of the set of subgroupoids of $(Q;f)$. As an illustration, consider the following

Example 2.7. Let $m \geq 3$ and let $V = RB$ be the variety of rectangular bands. Let Q be an arbitrary set and $f: V_n(Q) \to V_m(Q)$ be defined by $f(u) = a^m$ ($= a$ where a is a fixed element of Q). Then every subgroupoid of $(Q;f)$ is a strong subgroupoid as well. (Namely, P is a subgroupoid of $(Q;f)$ iff $a \in P$.)

3. Homomorphisms and congruences. First we note that in the case of $(V;n,m)$-groupoids the definition of a homomorphism can be restate as follows.

Proposition 3.1. If $(Q;f)$ and $(Q';f')$ are $(V;n,m)$-groupoids then a mapping $\phi:Q \to Q'$ is a homomorphism iff

$$\phi_m f = f'\phi_n \tag{3.1}$$

Proof. We have only to explain what are the meanings of ϕ_m, ϕ_n in (3.1). First, the mapping $\phi:Q \to Q'$ induces a unique homomorphism $\tilde{\phi}:V(Q) \to V(Q')$ such that $\tilde{\phi}(V_\alpha(Q))=V_\alpha(Q')$, for every $\alpha \geq 1$. Then we denote by $\phi_\alpha:V_\alpha(Q) \to V_\alpha(Q')$ the corresponding restriction of $\tilde{\phi}$. ✖

Proposition 3.2. If $\alpha:Q \to Q'$ is a bijective homomorphism then it is an isomorphism.

Proof. If $\phi:Q \to Q'$ is bijective, then $\tilde{\phi}$ is an isomorphism and $\phi_\alpha:V_\alpha(Q) \to V_\alpha(Q')$ is bijective as well, and $(\phi^{-1})_\alpha=(\phi_\alpha)^{-1}$. Then we have

$$(\phi^{-1})_m f'=(\phi^{-1})_m(f'\phi_n)(\phi_n)^{-1}=(\phi^{-1})_m\phi_m f(\phi_n)^{-1}=f(\phi^{-1}). \ ✖$$

We mentioned in Preliminaries that a homomorphic image of a subgroupoid is a subgroupoid, and that a complete inverse homomorphic image of a strong subgroupoid is a strong subgroupoid. The converse assertions are not true generally, as it show the following examples.

Example 3.3. Let $(Q;f)$ be a $(V;n,m)$-groupoid containing a subgroupoid P which is not a strong one, and let g be the restriction of f on P. Then P is a strong subgroupoid of $(P;g)$ and the embedding mapping from P into Q is a homomorphism such that P is a homomorphic image of a strong subgroupoid of $(P;g)$, but P is not strong in $(Q;f)$.

Example 3.4. Let V be the variety of commutative semigroups which satisfies the identity $x^2=y^2$, where x,y are different variables. If $Q=\{a,b,c\}$ and $Q'=\{\alpha,\beta\}$ then

$$V_4(Q) = \{a^4,a^3b,a^3c,b^3c\}, \qquad V_2(Q) = \{a^2,ab,ac,bc\},$$
$$V_4(Q') = \{\alpha^4,\alpha^3\beta\}, \qquad V_2(Q')= \{\alpha^2,\alpha\beta\}.$$

Define $(V;4,2)$-groupoids $(Q;f)$ and $(Q';f')$ by

$$f(u) = bc, \qquad f'(u') = \alpha^2,$$

for every $u\in V_4(Q)$, $u'\in V_4(Q')$.

Then the mappings

$$\phi = \begin{pmatrix} a & b & c \\ \alpha & \beta & \beta \end{pmatrix}, \qquad \psi = \begin{pmatrix} a & b & c \\ \alpha & \alpha & \alpha \end{pmatrix}$$

are homomorphisms from $(Q;f)$ into $(Q';f')$. The set $A'=\{\alpha\}$ is a subgroupoid of $(Q';f')$, but $A=\{a\}=\phi^{-1}(A')$ is not a subgroupoid of $(Q;f)$. Furthermore, $A=\{a\}$ is a generating subset of $(Q;f)$, and ϕ, ψ are different homomorphisms which extend the mapping $a \mapsto \alpha$ from A into Q'.

It is natural to define congruences as follows. Let $(Q;f)$ be a $(V;n,m)$-groupoid and ρ an equivalence on Q. We say that ρ is a congruence on $(Q;f)$ iff there is a homomorphism $\phi:(Q;f) \to (Q';f')$, where $(Q';f')$ is a $(V;n,m)$-groupoid, such that $\rho=\ker\phi$, i.e. $a\rho b \iff \phi(a) = \phi(b)$.

Let $(Q;f)$, $(Q';f')$, ϕ,ρ be as above. Then $P'=\phi(Q)$ is a subgroupoid of $(Q';f')$ and ϕ induces a unique surjective homomorphism $\psi:(Q;f) \to (P';g')$, where g' is the restriction of f' on P'. Moreover, we have $\ker\psi=\rho=\ker\phi$. Thus, we can assume that ϕ is surjective. Then $\bar{\phi}:\bar{a} \mapsto \phi(a)$ is bijective mapping from $\bar{Q}=Q/\rho$ onto $Q'=\phi(Q)$, where

$$\bar{a} = \{b\in Q \mid a\rho b\} = \{b\in Q \mid \phi(a) = \phi(b)\}.$$

This implies that if we define $\bar{f}:V_n(\bar{Q}) \to V_m(\bar{Q})$ by

$$\bar{f}(\bar{a}_1\cdot\ldots\cdot\bar{a}_n)=\bar{b}_1\cdot\ldots\cdot\bar{b}_m \iff f'(\phi(a_1)\cdot\ldots\cdot\phi(a_n))=\phi(b_1)\cdot\ldots\cdot\phi(b_m) \quad (3.2)$$

then we obtain a $(V;n,m)$-groupoid $(\bar{Q};\bar{f})$ such that $\bar{\phi}:\bar{a}\mapsto\phi(a)$ is an isomorphism from $(\bar{Q};\bar{f})$ onto $(Q';f')$.

Now we will give another characterization of congruences.

Proposition 3.5. Let $(Q;f)$ be a $(V;n,m)$-groupoid and ρ an equivalence on Q such that

$$f(a_1\cdot\ldots\cdot a_n)=b_1\cdot\ldots\cdot b_m,\quad f(c_1\cdot\ldots\cdot c_n)=d_1\cdot\ldots\cdot d_m \text{ in } (Q;f) \quad (3.3)$$

and

$$\bar{a}_1\cdot\ldots\cdot\bar{a}_n = \bar{c}_1\cdot\ldots\cdot\bar{c}_n \text{ in } V(\bar{Q}) \quad (3.4)$$

implies

$$\bar{b}_1\cdot\ldots\cdot\bar{b}_m = \bar{d}_1\cdot\ldots\cdot\bar{d}_m \text{ in } V(\bar{Q}), \quad (3.5)$$

where $\bar{Q}=Q/\rho$, $\bar{a}=\{b\in Q \mid a\rho b\}$.

Then ρ is a congruence on $(Q;f)$. Conversely, if ρ is a congruence on $(Q;f)$ then every implication $(3.3)\&(3.4)\Longrightarrow(3.5)$ holds.

Proof. Assume that $\phi:(Q;f)\to(Q';f')$ is a surjective homomorphism such that $\rho=\ker\phi$, and denote by $\bar{\phi}$ the corresponding isomorphism from $(\bar{Q};\bar{f})$ into $(Q';f')$.

If (3.3) holds in $(Q;f)$ then we have (in $(Q';f')$):

$$f'(\phi(a_1)\cdot\ldots\cdot\phi(a_n)) = \phi(b_1)\cdot\ldots\cdot\phi(b_m),$$
$$f'(\phi(c_1)\cdot\ldots\cdot\phi(c_n)) = \phi(d_1)\cdot\ldots\cdot\phi(d_m)$$

and therefore

$$\bar{f}(\bar{a}_1\cdot\ldots\cdot\bar{a}_n)=\bar{b}_1\cdot\ldots\cdot\bar{b}_m,\quad \bar{f}(\bar{c}_1\cdot\ldots\cdot\bar{c}_n)=\bar{d}_1\cdot\ldots\cdot\bar{d}_m$$

in $(\bar{Q};\bar{f})$. Assuming that (3.4) is satisfied, we obtain (3.5).

Conversely, assume that ρ is an equivalence on Q such that every implication $(3.3)\&(3.4)\Longrightarrow(3.5)$ holds.

If $a_1,\ldots,a_n\in Q$ and $f(a_1\cdot\ldots\cdot a_n)=b_1\cdot\ldots\cdot b_m$ in $(Q;f)$, then we define $\bar{f}(\bar{a}_1\cdot\ldots\cdot\bar{a}_n)$ by

$$\bar{f}(\bar{a}_1\cdot\ldots\cdot\bar{a}_n) = \bar{b}_1\cdot\ldots\cdot\bar{b}_m.$$

It follows from $(3.3)\&(3.4)\Longrightarrow(3.5)$ that $\bar{f}$ is well defined, i.e. we obtain a $(V;n,m)$-groupoid $(\bar{Q};\bar{f})$. Clearly, $\bar{\phi}:a\to\bar{a}$ is a homomorphism from $(Q;f)$ onto $(\bar{Q};\bar{f})$ and $\rho=\ker\bar{\phi}$, i.e. ρ is a congruence. ∎

(We remark that the above definition and Pr. 3.5 imply that the well known isomorphism theorems ([1]) holds.)

4. Induced $(W;n,m)$-groupoids. We assume here that W is a nontrivial subvariety of a variety V. Note that $W(Q)\in V$ for any nonempty set Q, which implies that there is a uniquely determined homomorphism $\pi:V(Q)\to W(Q)$ with the property $\pi(a)=a$ for all $a\in Q$. Moreover, for each positive integer p, $\pi(V_p(Q))=W_p(Q)$ and this implies that π induces a surjective mapping $\pi_p:V_p(Q)\to W_p(Q)$.

We say that a $(W;n,m)$-groupoid $(Q;g)$ is induced by a $(V;n,m)$-groupoid $(Q;f)$ iff the following diagram commutes:

$$\begin{array}{ccc} V_n(Q) & \xrightarrow{\ f\ } & V_m(Q) \\ \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_m} \\ W_n(Q) & \xrightarrow{\ g\ } & W_m(Q) \end{array}$$

An obvious consequence from this definition is

Proposition 4.1. If $(Q;f)$ is a $(V;n,m)$-groupoid then there exists at most one $(W;n,m)$-groupoid $(Q;g)$ which is induced by $(Q;f)$. Such a $(W;n,m)$-groupoid $(Q;g)$ do exist iff $(Q;f)$ satisfies the following condition:

$$(\forall u,v\in V(Q))(\pi_n(u) = \pi_n(v) \Longrightarrow \pi_m f(u) = \pi_m f(v)). \ ∎ \quad (4.1)$$

If a $(V;n,m)$-groupoid $(Q;f)$ satisfies (4.1) then we say that it <u>admits weakly</u> W. And $(Q;f)$ will be called a W-$(V;n,m)$-<u>groupoid</u> iff the following statement holds:

$$(\forall u,v \in V(Q))(\pi_n(u) = \pi_n(v) \Longrightarrow f(u) = f(v)). \qquad (4.1')$$

<u>Proposition 4.2</u>. A $(W;n,m)$-groupoid $(Q;g)$ is induced by at least one W-$(V;n,m)$-groupoid $(Q;f)$.

<u>Proof</u>. If $u \in W_n(Q)$ then $\pi_n^{-1}(u) \in V_n(Q)$, $\pi_m^{-1}(g(u)) \in V_m(Q)$.

If $f: V_n(Q) \to V_m(Q)$ is such that for every $x \in V_n(Q)$ we have $f(x) \in \pi_m^{-1}(g\pi_n(x))$ then we obtain a $(V;n,m)$-groupoid $(Q;f)$ which induces $(Q;g)$. Certainly, we can define f in such a way that it satisfies (4.1'). Namely, let $h: W_n(Q) \to V_m(Q)$ be such that $h(u) \in \pi_m^{-1} g(u)$ for every $u \in W_n(Q)$. Now, if we define $f: V_n(Q) \to V_m(Q)$ by $f = h\pi_n$, then we will obtain a W-$(V;n,m)$-groupoid $(Q;f)$ which induces $(Q;g)$. ✖

The following statements are also clear.

<u>Proposition 4.3</u>. Let $(Q;g)$ be a $(W;n,m)$-groupoid which is induced by a $(V;n,m)$-groupoid $(Q;f)$. Then:

(a) If P is a subgroupoid of $(Q;f)$ then P is a subgroupoid of $(Q;g)$.

(b) If ρ is a congruence on $(Q;f)$ then ρ is a congruence on $(Q;g)$. ✖

<u>Proposition 4.4</u>. Let $(Q;f)$ and $(Q';f')$ be $(V;n,m)$-groupoids and let $(Q;g)$, $(Q';g')$ be $(W;n,m)$-groupoids such that $(Q;g)$ is induced by $(Q;f)$ and $(Q';g')$ is induced by $(Q';f')$. If $\phi: Q \to Q'$ is a homomorphism from $(Q;f)$ into $(Q';f')$ then it is a homomorphism from $(Q;g)$ into $(Q';g')$ as well. ✖

The following example shows that Pr. <u>4.3 (a)</u>, in general, does not hold for strong subgroupoids.

<u>Example 4.5</u>. Let $Q=\{a,b\}$ and let $f: Q \to Q^3$ be defined by $f(a)=f(b)=(a,a,a)$. Define a mapping $g: RB(Q)=Q \to RB_3(Q)$ by $g(a)=g(b)=a$. Then $(Q;f)$ is a $(Sem;1,3)$-groupoid and $(Q;g)$ is a $(RB;1,3)$-groupoid induced by $(Q;f)$. $A=\{a\}$ is a strong subgroupoid of $(Q;f)$, but A is not a strong subgroupoid of $(Q;g)$.

R E F E R E N C E S

[1] P.M.Cohn: Universal Algebra, Harper & Row, New York (1965)

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ВЕКТОРСКО ВРЕДНОСНИ ГРУПОИДИ ИНДУЦИРАНИ ОД МНОГУОБРАЗИЈА ОД ПОЛУГРУПИ

Р е з и м е

Векторско вредносните групоиди индуцирани од полугрупи се разгледуваат во трудот [3]. Овде се разгледуваат истите прашања како и во претходно споменатиот труд со тоа што полугрупите се од дадено многуобразие од полугрупи. Се покажува дека некои резултати што не важат во општиот случај важат во вака извршената рестрикција.