ON FREE OBJECTS IN SOME CLASSES OF FINITE SUBSET STRUCTURES

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Introduction

The notions of groupoid and free groupoid are generalized in an unpublished paper [6]. One class of groupoids, generalized in this way, and existence of free objects in various subclasses are investigated in this paper.

Namely, every pair of covariant functors G, H from the category of sets (*Set*) into itself determines a class of (G,H)-groupoids, defined as follows.

An ordered pair $(Q;f)$, where Q is a nonempty set and f a mapping from $G(Q)$ into $H(Q)$, is said to be a (G,H)-*groupoid*. Special classes of (G,H)-groupoids are, for example, the class of (binary) groupoids, or, more generally, the class of (n,m)-groupoids, where $G(Q) = Q^n$ and $H(Q) = Q^m$.

If $(Q;f)$ and $(Q';f')$ are (G,H)-groupoids, a mapping $\varphi : Q \to Q'$ is said to be a *homomorphism*, if the following diagram commutes.

$$\begin{tikzcd}
Q \arrow[r, "G"] \arrow[d, "\varphi"] & G(Q) \arrow[r, "H"] \arrow[d, "G(\varphi)"] & H(Q) \arrow[d, "H(\varphi)"] \\
Q' \arrow[r] & G(Q') \arrow[r] & H(Q')
\end{tikzcd}$$

Let $(Q;f)$ be a (G,H)-groupoid, B, P nonempty subsets of Q. P is said to be a *subgroupoid* of $(Q;f)$ iff $f(G(P)) \subseteq H(P)$, and B *generates* $(Q;f)$ iff Q is the unique subgroupoid of $(Q;f)$ containing B.

Having in mind all said above, we obtain that (G,H)-groupoids form a category, where morphisms are the homomorphisms.

The following definition of free (G,H)-groupoids in a class $\mathcal{C}$ of (G,H)-groupoids is formally the same as the usual one. Namely:

$\mathbf{Q} = (Q;f) \in \mathcal{C}$ is a *free* (G,H)-groupoid with a basis B in the class $\mathcal{C}$ iff the following conditions are satisfied:

(i) B generates $(Q;f)$;

(ii) for any (G,H)-groupoid $(Q',f') \in \mathcal{C}$ and any mapping $\lambda : Q \to Q'$, there exists a homomorphism $\varphi : (Q;f) \to (Q';f')$ which extends λ.

The following problems for a class $\mathcal{C}$ of (G,H)-groupoids appear to be interesting for investigation.

I. Is any (nonempty) set B a basis of a free (G,H)-groupoid in $\mathcal{C}$?

If the answer is positive, then:

II. Is the homomorphism φ uniquely determined by λ?

III. Are two free (G,H)-groupoids with a same basis isomorphic?

IV. Give a description of a free (G,H)-groupoid with a given basis B in the class $\mathcal{C}$.

It is shown in [3], [8] and [7], that the answer to each of the questions above is, in general, negative.

In this paper we consider a special class of (G,H)-groupoids, namely the class of (F,F)-groupoids, where for each nonempty set Q, $F(Q)$ is the family of all nonempty finite subsets of Q. All the mentioned problems are investigated for various classes of "associative" (F,F)-groupoids.

Further on, instead of (F,F)-groupoid $(Q;f)$ we will say *object*, using the fact that it is an object in the category of (F,F)-groupoids, but *we will not use any category theory approach in investigating* (F,F)-groupoids.

1. Associative objects

Let Q be a nonempty set, and denote by $F(Q)$ the collection of finite nonempty subsets of Q. We say that $(Q;f)$ is an *object* if f is a transformation of $F(Q)$, i.e. a mapping from $F(Q)$ into $F(Q)$. If $(Q;f)$ and $(Q';f')$ are two objects, then a mapping

$\varphi : Q \to Q'$ is a *homomorphism* from $(Q;f)$ into $(Q';f')$ iff $F(\varphi)f = f'F(\varphi)$, where $F(\varphi): F(Q) \to F(Q')$ is the corresponding mapping induced by φ. We will usually write φ instead of $F(\varphi)$.

If $(Q;f)$ is an object and P a nonempty subset of Q such that $X \in F(P) \Rightarrow f(X) \in F(P)$, then we say that P is a *subobject* of $(Q;f)$.

The following statements are clear.

Proposition 1.1. *A nonempty intersection of subobjects of an object $(Q;f)$ is also a subobject of $(Q;f)$.* □

Proposition 1.2. *If $\varphi : Q \to Q'$ is a bijective homomorphism from $(Q;f)$ into $(Q';f')$ then $\varphi^{-1}: Q' \to Q$ is a homomorphism from $(Q';f')$ into $(Q;f)$.* □

(In this case we say that φ is an *isomorphism*.)

Proposition 1.3. *A homomorphic image of a subobject is a subobject, and a nonempty inverse homomorphic image of a subobject is a subobject as well.* □

From Proposition 1.1 it follows that if $(Q;f)$ is an object then every nonempty subset B of Q generates a uniquely defined subobject $\langle B \rangle$ of $(Q;f)$.

An object $(Q;f)$ is said to be *associative* if:

$$(\forall X,Y \in F(Q))\ f(f(X) \cup Y) = f(X \cup Y). \tag{1.1}$$

Proposition 1.4. *If $(Q;f)$ is an associative object and $*$ is a (binary) operation on $F(Q)$ defined by*

$$(\forall X,Y \in F(Q))\ X * Y = f(X \cup Y) \tag{1.2}$$

then

(i) *$(F(Q);*)$ is a commutative semigroup,*

(ii) *$(\forall X,Y,Z \in F(Q))\ X*(Y \cup Z) = X*Y*Z$,*

(iii) *$(\forall X \in F(Q))\ f(X) = X*X$.*

Conversely, if $(F(Q);*)$ satisfies the conditions (i), (ii) *and $f : F(Q) \to F(Q)$ is defined by* (iii), *an associative object $(Q;f)$ is obtained, such that* (1.2) *is satisfied.* □

(We say that $(Q;f)$ and $(F(Q);*)$ are *associated*.)

Note that if $(Q;f)$ is an associative object and $*$ is defined by (1.2), then

$$f(\{a_1, \dots, a_n\}) = a_1 * a_2 * \dots * a_n. \tag{1.3}$$

(Here we do not make any distinction between $\{a\}$ and a, when $a \in Q$.)

Proposition 1.5. *Let $(Q;f)$ and $(Q';f')$ be associative objects, and $(F(Q);*)$ and $(F(Q');*)$ the associated semigroups. Then*

(i) *P is a subobject of $(Q;f)$ iff $F(P)$ is a subsemigroup of $(F(Q);*)$.*

(ii) *$\varphi : Q \to Q'$ is a homomorphism iff $F(\varphi): F(Q) \to F(Q')$ is a homomorphism, as well.* □

An object $(Q;f)$ is said to be an *m-object* if $f(F(Q)) \subseteq F_m(Q)$, where $F_m(Q) = \{A \in F(Q) \mid |A| \le m\}$.

Proposition 1.6. *An associative object $(Q;f)$ is an associative m-object iff*

$$(\forall X,Y \in F(Q))\ X*Y \in F_m(Q).$$ □

(Then $F_m(Q)$ is an *ideal* in $(F(Q);*)$.)

Proposition 1.7. *The class of associative objects (m-objects) is hereditary and closed under homomorphic images.* □

Now we are ready to define a special class of associative m-objects, for every positive integer m. Namely, we say that $(Q;f)$ is an *m-semilattice* iff $(Q;f)$ is an associative m-object such that the corresponding semigroup $(F_m(Q);*)$ is a semilattice.

Having in mind that $X*X = f(X \cup X) = f(X)$, we obtain the following characterization of m-semilattices.

Proposition 1.8. *An object $(Q;f)$ is an m-semilattice iff it is an associative m-object with the following property:*

$$(\forall X \in F_m(Q))\ f(X) = X, \tag{1.4}$$

i.e., f is a retract. □

Proposition 1.9. *The class of m-semilattices is hereditary and closed under homomorphic images.* □

In the special case, when $m = 1$, we have the following

Proposition 1.10. *$(Q;f)$ is a 1-semilattice iff there is a (uniquely defined) semilattice $(Q;*)$ such that (1.3) holds. Then the following statements are also satisfied:*

(i) *P is a subobject of $(Q;f)$ iff P is a subsemilattice of $(Q;*)$.*

(ii) *B is a generating subset of $(Q;f)$ iff B is a generating subset of $(Q;*)$.*

(iii) *Let $(Q;f)$ and $(Q';f')$ be 1-semilattices. A mapping $\varphi : Q \to Q'$ is a homomorphism from $(Q;f)$ into $(Q';f')$ iff it is a homomorphism from $(Q;*)$ into $(Q';*')$.* □

Further on we will assume that $m \geq 2$.

From Proposition 1.8 it follows that if $|Q| \leq m$, and if $(Q;f)$ is an m-semilattice, then $f(X) = X$, for every $X \in F(Q)$.

Proposition 1.11. *If $(Q;f)$ is an m-semilattice and if $X \in F(Q)$, $|X| > m$, then $|f(X)| = m$.*

Proof. Assume that $|f(X)| < m$. Then there exists an $a \in X \setminus f(X)$, and therefore we would have: $f(X) \cup a = f(f(X) \cup a) = f(X \cup a) = f(X)$. □

Example 1.12. Let Q be a set with at least m distinct elements, and let $A \in F(Q)$ be such that $|A| = m$. Then, by

$$f(X) = \begin{cases} A & \text{if } |X| > m \\ X & \text{if } |X| \leq m \end{cases} \tag{1.5}$$

an m-semilattice is defined, and the corresponding associated semigroup $(F(Q);*)$ is defined by

$$X*Y = \begin{cases} A & \text{if } |X \cup Y| > m \\ X \cup Y & \text{if } |X \cup Y| \leq m. \end{cases}$$

We say that $(Q;f)$ is a *constant* m-semilattice.

Example 1.13. Let $Q = \{a,b,c,d\}$ and let $f : F(Q) \to F(Q)$ be defined as follows:

$$f(\{a,b,c\}) = \{a,b\}, \qquad |X| \leq 2 \Rightarrow f(X) = X, \qquad |X| \geq 3,\ X \neq \{a,b,c\} \Rightarrow f(X) = \{c,d\}.$$

Then we obtain a non-constant 2-semilattice $(Q;f)$, and the corresponding semigroup $(Q;*)$ is defined by

$$X*Y = X \cup Y \text{ if } |X \cup Y| \leq 2, \qquad X*Y = \begin{cases} \{a,b\} & \text{if } X \cup Y = \{a,b,c\} \\ \{c,d\} & \text{if } |X \cup Y| \geq 3,\ X \cup Y \neq \{a,b,c\}. \end{cases}$$

2. Free associative m-objects

Let $\mathcal{C}$ be a class of objects.

An object $(Q;f) \in \mathcal{C}$ is said to be a *free object in $\mathcal{C}$ with a basis B* iff the following conditions are satisfied:

(i) B is a generating subset of $(Q;f)$;

(ii) for every object $(Q';f') \in \mathcal{C}$ and every mapping $\lambda : B \to Q'$ there is a homomorphism from $(Q;f)$ into $(Q';f')$, which is an extension of λ.

The following results are shown in [3].

Proposition 2.1. *There does not exist a free object in the class of all objects.*

Proof. Let $(Q;f)$ be an object and B a nonempty subset of Q, and let $f(a) = A$, where $a \in B$, $A \in F(Q)$. Let P be a nonempty set such that $B \subseteq P$, $|A| < |P|$, and let $(P;g)$ be an object such that $(\forall X \in F(P))\, g(X) = C$, where $|A| < |C|$, and C is a given element of $F(P)$. Then the embedding mapping $\lambda : B \to P$ can not be extended to a homomorphism φ from $(Q;f)$ into $(P;g)$. □

Proposition 2.2 ([3, Prop. 3.12]). *Let B be a nonempty set, $B \cap \mathbb{N} = \emptyset$ and let a sequence of sets $\{C_\alpha | \alpha \geq 0\}$ be defined by*

$$C_0 = B, \qquad C_{p+1} = C_p \cup (\mathbb{N}_m \times F(C_p)), \tag{2.1}$$

where $\mathbb{N}_m = \{1,2,\ldots,m\}$, and m is a given positive integer. If $S_B = \bigcup\{C_p | p \geq 0\}$, and if an object $(S_B;f)$ is defined by

$$(\forall X \in F(S_B))\ f(X) = \{(1,X),\ldots,(m,X)\}, \tag{2.2}$$

then $(S_B;f)$ is a free m-object with a basis B in the class of all m-objects.

Moreover, every endomorphism φ of $(S_B;f)$, such that $(\forall b \in B)\ \varphi(b) = b$ is an automorphism, and the set of all such automorphisms is infinite.

Every free m-object $(Q;f')$ with a basis B is isomorphic to $(S_B;f)$. □

Proposition 2.3. *The class of associative objects does not contain free members.*

Proof. The object $(P;g)$ from the proof of 2.1 is an associative object. □

Now we are going to show that there do exist free objects in the class of associative m-objects, and in the class of m-semilattices, as well.

For that purpose we will choose a special subset R_B of the set S_B, which was defined in Proposition 2.2.

Let $y\in S_B$. If p is the least nonnegative integer such that $y\in C_p$, then we write $\chi(y)=p$ and say that p is the *hierarchy* of y. The hierarchy $\chi(Y)$ of a set $Y\in F(S_B)$ is the number $\max\{\chi(y) \mid y\in Y\}$.

We will next define a relation α in $F(S_B)$.

(a) If $X,Y\in F(S_B)$, then: $X\alpha Y \Leftrightarrow X\alpha y$, for each $y\in Y$.

Thus, it remains to define the meaning of $X\alpha y$, for $X\in F(S_B)$, and $y\in S_B$, and we will define this relation by induction on the hierarchy of y. (Here we use the notation u for the set $\{u\}$.)

First:

(b) $\chi(y)=0 \Rightarrow (X\alpha y \Leftrightarrow y\in X)$.

Assume that $u=(i,Y)\in S_B$, $\chi(u)=t\geq 1$, and we have a procedure to determine whether $X\alpha y$, for every $X\in F(S_B)$, $y\in S_B$, such that $\chi(y)<t$. Then $X\alpha u$, iff at least one of the following conditions is satisfied:

(c_1) $u\in X$, (c_2) $X\alpha y$, for every $y\in Y$.

By induction on hierarchy, it can be easily seen that α is a well defined relation in $F(S_B)$. (If $X\alpha Y$, we say that "X *absorbs* Y".)

Proposition 2.4. *If $X,Z\in F(S_B)$, $y\in S_B$ are such that $X\subseteq Z$ and $X\alpha y$, then $Z\alpha y$.*

Proof. In the case $y\in X$ the conclusion is trivial. Assume that $y=(i,U)$, and that $Z\alpha u$, for every $u\in U$. Therefore, using induction on hierarchy, we obtain $Z\alpha y$. □

Proposition 2.5. *If $X\in F(S_B)$, $y\in S_B\backslash X$ and $X\alpha y$, then there exists a subset Z of X, such that $Z\alpha y$ and $\chi(Z)<\chi(y)$.*

Proof. It is clear that $y\notin B$. Therefore $y=(i,U)\in C_{p+1}\backslash C_p$, for some $p\geq 0$, $i\in\mathbb{N}_m$, $U\in F(C_p)$, and we have $X\alpha u$, for every $u\in U$. By induction on hierarchy, we can assume that for every $u\in U$, there is a $Z_u\in X$, such that $Z_u\alpha u$, $\chi(Z_u)\leq\chi(u)\leq p$. If $Z=\bigcup\{Z_u \mid u\in U\}$, then by Proposition 2.4, we obtain that $Z\alpha u$ for every $u\in U$, and therefore $Z\alpha y$. Moreover, we have $\chi(Z)\leq\chi(U)<\chi(y)$. □

Now we will define a subset R_B of S_B (we will say that R_B is the set of irreducible elements of S_B) as follows:

1) $B\subseteq R_B$;

2) $u=(i,Y)\in R_B$ iff the following conditions are satisfied:

2.1) $Y\in F(R_B)$,

2.2) there does not exist a $z\in Y$, such that $(Y\backslash z)\alpha z$,

2.3) Y does not contain a subset of the form $\{(1,Z),(2,Z),\dots,(m,Z)\}$.

An $X\in F(R_B)$ is said to be *reducible* iff it satisfies the following conditions:

2.2') there exists a $z\in X$ such that $(X\backslash z)\alpha z$,

or

2.3') there exists a subset of X of the form $\{(1,Z),(2,Z),\dots,(m,Z)\}$.

$X\subseteq R_B$ is *irreducible* iff it is not reducible.

The next step is to define an associative object on R_B. For that purpose we need a definition of *norm* $\|X\|$, $X\in F(R_B)$. It is defined by induction on hierarchy, in the following way:

3.1) $\|X\|=0 \Leftrightarrow X\subseteq B$;

3.2) $\|(i,X)\|=1+\|X\|$;

3.3) if $X=\{x_1,\dots,x_n\}$, $|X|=n$, then $\|X\|=\|x_1\|+\|x_2\|+\dots+\|x_n\|$.

Now we will define an associative object $(R_B;g)$ as follows:

(i) If $X\in F(R_B)$ is irreducible, then $g(X)=X$.

Assume now that $X\in F(R_B)$ is reducible and for every $Y\in F(R_B)$, such that $\|Y\|<\|X\|$, an irreducible set $g(Y)\in F(R_B)$ is well defined and the following relation holds:

$$g(Y)\neq Y \Leftrightarrow \|g(Y)\|<\|Y\|. \tag{2.3}$$

Consider, first, the case when 2.2') is satisfied, and let

$$X=X_{p_1}\cup\dots\cup X_{p_k}, \tag{2.4}$$

where $p_1<\dots<p_k$, and $x\in X_{p_\nu} \Leftrightarrow \chi(X)=p_\nu$.

By Proposition 2.5, X_{p_1} does not satisfy 2.2'). Let s be the greatest number such that

$$X' = X_{p_1} \cup \ldots \cup X_{p_s},$$

does not satisfy 2.2'). Then $1 \leq s < k$. Denote by Z the set of all $z \in X \setminus X'$, such that $X' \alpha z$, and let $Y = X \setminus Z$. Then we have $Z \neq \emptyset$, $Z \cap Y = \emptyset$ and $\|Y\| < \|X\|$. Therefore $g(Y) \in F(R_B)$ is a well defined irreducible set, and now we define $g(X)$ by:

(ii) $g(X) = g(Y)$.

We have $\|g(X)\| = \|g(Y)\| \leq \|Y\| < \|X\|$, i. e. (2.3) holds.

Finally, assume that X does not satisfy 2.2'). Then 2.3') holds, and therefore X has the form

$$X = X' \cup \{(1, Z_1), \ldots, (m, Z_1), \ldots, (1, Z_k), \ldots, (m, Z_k)\},$$

where $X' = \emptyset$ or X' is irreducible and $\nu \neq \lambda \Rightarrow Z_\nu \neq Z_\lambda$. Now we have $\|X' \cup Z_1 \cup \ldots \cup Z_k\| < \|X\|$, and thus $g(X)$ can be defined by:

(iii) $g(X) = g(X' \cup Z_1 \cup \ldots \cup Z_k)$.

In this case, we also have

$$\|g(X)\| = \|g(X' \cup Z_1 \cup \ldots \cup Z_k)\| \leq \|X' \cup Z_1 \cup \ldots \cup Z_k\| < \|X\|.$$

Therefore $g : F(R_B) \to F(R_B)$ is a well defined mapping, such that (2.3) holds for every $Y \in F(R_B)$.

Proposition 2.6. *If $y \in X \in F(R_B)$ and $(X \setminus y) \alpha y$, then $g(X) = g(X \setminus y)$.*

Proof. In the recursive definition, $g(X)$ is defined by (ii). If $y \notin Z$ (Z is as in (ii) of the definition of g), then by induction on norm we have:

$$g(X) = g(X \setminus Z) = g((X \setminus Z) \setminus y) = g(X \setminus y).$$

If $y \in Z$, then

$$g(X) = g(X \setminus Z) = g((X \setminus Z) \cup (Z \setminus y)) = g(X \setminus y). \ \square$$

Proposition 2.7. *If $X = X' \cup \{(1, Y), \ldots, (m, Y)\} \in F(R_B)$, where $X' = \emptyset$ or $X' \in F(R_B)$, then $g(X) = g(X' \cup Y)$.*

Proof. Assume, first, that $(i, Y) \in X'$, for some $i \in \mathbb{N}_m$. Then we obtain the equation $g(X) = g(X' \cup Y)$ by induction on $\|X'\|$. Thus we may assume that the above union is disjoint.

If $(X' \setminus (i, Y)) \alpha (i, Y)$ for some $i \in \mathbb{N}_m$, then we have $X' \alpha (i, Y)$, for every $i \in \mathbb{N}_m$, and by Proposition 2.6, we obtain: $g(X) = g(X') = g(X' \cup Y)$.

If $(X' \setminus u) \alpha u$, for some $u \in X'$, then we obtain $g(X) = g(X' \cup Y)$, again by Proposition 2.6.

It remains to consider the case when X does not satisfy 2.2'). Then, if $X' = \emptyset$ or X' is irreducible, the equation $g(X) = g(X' \cup Y)$ follows by (iii); and if

$$X' = X'' \cup \{(1, Z_1), \ldots, (m, Z_1), \ldots, (1, Z_r), \ldots, (m, Z_r)\},$$

where $X'' = \emptyset$ or X'' is irreducible, then we have:

$$g(X) = g(X'' \cup Z_1 \cup \ldots \cup Z_r \cup \{(1, Y), \ldots, (m, Y)) = g(X'' \cup Z_1 \cup \ldots \cup Z_r \cup Y = g(X' \cup Y). \quad \square$$

Now we can show the following:

Proposition 2.8. *$(R_B; g)$ is an associative object.*

Proof. If X is irreducible, then $g(X \cup Y) = g(g(X) \cup Y)$. In the case when X is reducible, then by Proposition 2.6 or Proposition 2.7 and an induction on norm we obtain that $g(X \cup Y) = g(g(X) \cup Y)$. $\square$

Now we are ready to give a construction of a free m-object with a basis B.

First we define an m-object $(R_B; f)$ by

$$f(X) = \{(1, g(X)), \ldots, (m, g(X))\}, \tag{2.5}$$

for every $X \in F(R_B)$.

Proposition 2.9. *$(R_B; f)$ is an associative free m-object with a basis B.*

Proof. It is clear that $(R_B; f)$ is an m-object, and, moreover, we have:

$$f(f(X) \cup Y) = f(\{(1, g(X)), \ldots, (m, g(X))\} \cup Y) = \{(i, g(\{(1, g(X)), \ldots, (m, g(X))\} \cup Y)) \mid i \in \mathbb{N}_m\} =$$
$$= \{(i, g(g(X) \cup Y)) \mid i \in \mathbb{N}_m\} = \{(i, g(X \cup Y)) \mid i \in \mathbb{N}_m\} = f(X \cup Y),$$

i. e., $(R_B; f)$ *is* associative.

Let P be a subobject of $(R_B; f)$ such that $B \subseteq P$. By induction on hierarchy we will show that $P = R_B$. Assume that $\{u \in R_B \mid \chi(u) \leq p\} \subseteq P$, and let $y \in R_B$ be such that $\chi(y) = p+1$. Then $y = (i, Y)$ for some $i \in \mathbb{N}_m$, and $Y \in F(R_B)$, where Y is irreducible and $\chi(Y) = p$. Thus

$Y \subseteq P$, and $g(Y) = Y$. From (2.5) we obtain:

$$f(Y) = \{(1,Y),(2,Y),\ldots,(i,Y),\ldots,(m,Y)\} \subseteq P,$$

and therefore $(i,Y) \in P$. This implies that $P = R_B$, i. e. that B is a generating subset of $(R_B;f)$

Let $(Q';f')$ be an associative m-object and $\lambda : B \to Q'$ an arbitrary mapping. We will show that there is a homomorphism $\varphi : R_B \to Q'$, from $(R_B;f)$ into $(Q';f')$ which is an extension of λ.

Denote by D_p the set of all elements x of R_B such that $\chi(x) \le p$, and assume that for every $r \le p$, $\varphi_r : D_r \to Q'$ is a mapping with the following properties:

(a) $\varphi_0 = \lambda$;

(b) φ_r is an extension of φ_{r-1};

(c) $\varphi_r(f(X)) = f'(\varphi_r(X))$,

for every $X \in F(D_r)$ and $r \in \mathbb{N}_p$.

Define a mapping $\varphi_{p+1} : D_{p+1} \to Q'$ as follows. First $\varphi_{p+1}(u) = \varphi_p(u)$, for every $u \in D_p$. Let $u = (i,X) \in D_{p+1}$, i. e. $X \in F(D_p)$ is such that $\chi(X) = p$. Then $\varphi_p(X) \in F(Q')$, $|f'(\varphi_p(X))| \le m$, and

$$f(X) = \{(1,X),(2,X),\ldots,(m,X)\}.$$

Therefore, there is a surjective mapping $\psi_X : f(X) \to f'(\varphi_p(X))$. If we choose such a surjective mapping ψ_X for every $X \in F(D_p)$, which has a hierarchy p, and if we put:

$$\varphi_{p+1}(i,X) = \psi_X(i,X),$$

then we obtain a mapping $\varphi_{p+1} : D_{p+1} \to Q'$ which is an extension of φ_p, and, moreover, (c) is true for $r = p+1$, as well.

In such a way we will obtain a collection of mappings $\{\varphi_p : D_p \to Q' \mid p \ge 0\}$, such that (a), (b) and (c) are satisfied for every positive integer r. If $\varphi = \bigcup\{\varphi_p \mid p \ge 0\}$, we obtain a homomorphism from $(R_B;f)$ into $(Q';f')$ which is an extension of λ. □

Proposition 2.10. *Every endomorphism of $(R_B;f)$ which is an extension of the embedding mapping from B into R_B is an automorphism, and, moreover (in the case $m \ge 2$) the set of such automorphisms is infinite.*

Proof. Let φ be an endomorphism of $(R_B;f)$ which is an extension of the embedding from B into R_B. By induction on p we will show that for every $p \ge 0$, φ induces a permutation η_p of $T_p = \{u \in R_B \mid \chi(u) = p\}$. First η_0 is the identity permutation on $T_0 = B$. Assume that if we put $(\forall u \in T_p)\ \varphi(u) = \eta_p(u)$, then we obtain a permutation of T_p. Let $x \in T_{p+1} \setminus T_p$. Then $x = (i,X)$, for some $i \in \mathbb{N}_m$, and $X \in F(R_B)$, such that $\chi(X) = p$, and moreover, $\chi(\varphi(X)) = p$. Then

$$f(X) = \{(1,X),(2,X),\ldots,(m,X)\}$$

and this implies

$$\{\varphi(1,X),\varphi(2,X),\ldots,\varphi(m,X)\} = \varphi(f(X)) = f'(\varphi(X)) = \{(1,\varphi(X)),(2,\varphi(X)),\ldots,(m,\varphi(X))\},$$

i. e., there is a permutation $\tau \in S_m$, such that $\varphi(\nu,X) = (\tau(\nu),\varphi(X))$. Hence, φ induces a permutation η_{p+1} of T_{p+1}, as well. This completes the proof that φ is a permutation of R_B, i. e. an automorphism of $(R_B;f)$.

The fact that there exist infinitely many such automorphisms follows from the last part of the proof of the preceding proposition. □

Proposition 2.11. *Every free associative m-object with a basis B is isomorphic to $(R_B;f)$.*

Proof. Assume that $(T;f')$ is an arbitrary free associative m-object with a basis B. Then there exist homomorphisms $\varphi : R_B \to T$, and $\eta : T \to R_B$, such that $(\forall b \in B)\ \eta\varphi(b) = b$, and therefore $\eta\varphi$ is a permutation of R_B, which implies that φ is an injective mapping. Then $P = \varphi(R_B)$ is a subobject of $(T;f')$ which is generated by B, and thus $P = T$, whence we obtain that φ is an isomorphism. □

The object $(R_B;f)$ is not an m-semilattice, because $f(X) \neq X$, for every $X \in F_m(R_B)$.

Below we will give a construction of free m-semilattices.

First we note that:

Proposition 2.12. *If $|B| \le m$, then the trivial m-semilattice on B is a free m-semilattice with a basis B.* □

Assume now that $|B| > m$, and define a subset L_B of R_B in the following way:

$L_0 = B$; $L_{p+1} = \{(i,X) \mid i \in \mathbb{N}_m,\ X \in F(L_p),\ |X| > m\}$,

$L_B = \bigcup\{L_p \mid p \geq 0\}$.

Define an object $(L_B;h)$ as follows. If $X \in F(L_p)$, then:

$$h(X) = \begin{cases} X, & \text{if } |X| \leq m, \\ \{(1,g(X)),(2,g(X)),\dots,(m,g(X))\}, & \text{if } |X| > m. \end{cases}$$

As a corollary of the Propositions 2.9, 2.10, 2.11 and the fact that if $X \in F(L_p)$ and $|X| > m$, then $|g(X)| > m$, we obtain the following statement:

Proposition 2.13. (a) *$(L_B;h)$ is a free m-semilattice with a basis B.*

(b) *Every endomorphism of $(L_B;h)$ that is an extension of the embedding mapping from B into L_B is an automorphism.*

(c) *Every free m-semilattice with a basis B is isomorphic to $(L_B;h)$.* □

An associative m-object $(Q;f)$ is called an *(m,n)-semilattice* iff $n \leq m$ and $(\forall X \in F_n(Q))\ f(X) = X$. Thus the class of (m,m)-semilattices coincides with the class of m-semilattices.

Assume now that $|B| > n$, where $1 \leq n \leq m$, and define a subset L_B^n of R_B as follows:

$M_0 = B$, $M_{p+1} = M_p \cup (\mathbb{N}_m \times \{X \in F(M_p) \mid |X| > n\})$,

$L_B^n = \bigcup\{M_p \mid p \geq 0\}$.

Consider the following m-object $(L_B^n;l)$. If $X \in F(L_B^n)$, then

$$l(X) = \begin{cases} X, & \text{if } |X| \leq n, \\ \{(1,g(X)),(2,g(X)),\dots,(m,g(X))\}, & \text{if } |X| > n. \end{cases}$$

Proposition 2.14. (a) *$(L_B^n;l)$ is a free object with a basis B in the class of (m,n)-semilattices.*

(b) *Every endomorphism of $(L_B^n;l)$ which is an extension of the embedding mapping from B into L_B^n is an automorphism.*

(c) *Every free (m,n)-semilattice with a basis B is isomorphic to $(L_B^n;l)$.* □

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