ON POLY-ALGEBRAS

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Abstract

In this paper poly-algebras of a given type $\mathcal{F}$ are considered. Namely, $\emptyset \neq \mathcal{F} = \cup\{\mathcal{F}_n : n \geq 0\}$ is a disjoint union, and a poly$-\mathcal{F}-$algebra $\mathcal{A}$ with a carrier $A \neq \emptyset$ is a mapping $\mathcal{A} : f \to f^{\mathcal{A}}$ such that $f^{\mathcal{A}} : A^n \to \mathcal{P}(A)$, for every $f \in \mathcal{F}_n, n \geq 0$. Subalgebras are defined in the usual way, but, there are considered three kinds of homomorphisms, which implies three kinds of "freeness". Several results about subalgebras, homomorphisms and free objects in different classes of poly-algebras are given.

1. Poly-algebras

Let A be a non-empty set. Any mapping from A^n into the family $\mathcal{P}(A)$ of all subsets of A is called a *poly-n-operation on A*. Intuitively, a poly-algebra is a non-empty set with some poly-operations. Precisely, let $\mathcal{F}$ be a (non-empty) disjoint union of some sets $\{\mathcal{F}_n : n \in \mathbf{N}\}$, (i.e. sets of $n-$ary functional symbols). The set $\mathcal{F}$ we call a *type (of poly-operations).* We assume that $\mathcal{F} \neq \emptyset$ but it can be $\mathcal{F}_n = \emptyset$ for some $n \in \mathbf{N}$. We say that $\mathcal{A}$ is a *poly-algebra of type* $\mathcal{F}$ (or *poly* $-$ $\mathcal{F}$ $-$ *algebra*) *with a carrier A* if $\mathcal{A} : f \mapsto f^{\mathcal{A}}$ is a mapping from $\mathcal{F}$ into the family of poly-operations on A, such that if $f \in \mathcal{F}_n$ then $f^{\mathcal{A}}$ is a poly$-n-$operation on A. The family of poly$-\mathcal{F}-$algebras with the carrier A will be denoted by $\mathcal{F}(A)$. In this paper we shall assume that $\mathcal{F}$ is a fixed type of poly-operation.

If $\mathcal{A} \in \mathcal{F}(\mathcal{A})$, $f \in \mathcal{F}_n$, $n \geq 1$, and $Q \subseteq A^n$, then $f^{\mathcal{A}}(Q)$ has the usual meaning, i.e. $f^{\mathcal{A}}(Q) = \cup\{f^{\mathcal{A}}(x) : x \in Q\}$.

If we do not make any distinction between an element $a \in A$ and the corresponding one element subset $\{a\}$ of A, then every usual $n-$ary operation $f : A^n \to A$ can be considered as a special poly-operation. In this sense every algebra is a special poly-algebra.

Partial algebras are also special poly-algebras. Namely, we say that $\mathcal{A} \in \mathcal{F}(A)$ is a partial $\mathcal{F}-$algebra on $\mathcal{A}$ if $|f(x)| \leq 1$, for every $f \in \mathcal{F}_n, n \geq 0, x \in A^n$. Clearly, there is not an essential difference between this definition of partial algebras and the usual one (see for example [5]).

Also, there is a bijection poly-operations on a set A and $n+1-$ary relations of A (i.e. subsets of A^{n+1}). Namely, if $R \subseteq A^{n+1}$ then the corresponding poly-operation f^R is given by

$$y \in f^R(x_1, \ldots, x_n) \iff (x_1, \ldots, x_n, y) \in R.$$

(Note that a poly-0-operation on A is a subset of A.) This bijection implies that every poly-algebra can be interpreted as a relational structure and conversely. It should also be noted that a poly-algebra $\mathcal{A} \in \mathcal{F}(A)$ induces a usual (universal) algebra $\mathcal{C}m(\mathcal{A})$. $\mathcal{C}m(\mathcal{A})$ is a Boolean algebra with operators. The carrier of $\mathcal{C}m(\mathcal{A})$ is $\mathcal{P}(A)$, and the operators $\{f^{\mathcal{C}m(\mathcal{A})} : f \in \mathcal{F}\}$ are defined in the following way: if $f \in \mathcal{F}_n$ then

$$f^{\mathcal{C}m(\mathcal{A})}(M_1, \ldots, M_n) = \cup\{f^{\mathcal{A}}(x_1, \ldots, x_n) : x_k \in M_k\}.$$

The algebra $\mathcal{C}m(\mathcal{A})$ is called *the complex algebra of* $\mathcal{A}$. These complex algebras turn out to be very useful in studying various representability questions in the case of Boolean algebras with operators (see for example [4], [6], [7]). We also note that if the type $\mathcal{F}$ contains only one binary operation symbol, then a poly$-\mathcal{F}-$algebra is usually called a *multi-grupoid* ([1], II.7, p. 41).

We can define a natural ordering $\leq$ in $\mathcal{F}(A)$ in the following way:

$$\mathcal{A} \leq \mathcal{A}_1 \text{ iff } f^{\mathcal{A}}(x) \subseteq f^{\mathcal{A}_1}(x), \text{ for all } n \in \mathbf{N}, f \in \mathcal{F}_n, x \in A^n.$$

It is not hard to see that $(\mathcal{F}(A), \leq)$ is a complete boolean lattice. The least element in this lattice will be denoted by $o(A)$ (the so called *singular poly-algebra on* A) and the largest by $e(A)$ (*the unit poly-algebra on* A). Thus, for all $n \in \mathbf{N}, f \in \mathcal{F}_n, x \in A^n$ we have

$$f^{o(A)}(x) = \emptyset, \text{ and } f^{e(A)}(x) = A.$$

We shall say that $\mathcal{A} \in \mathcal{F}(A)$ is *regular* iff $f^{\mathcal{A}}(x) \neq \emptyset$, for all $n \in \mathbf{N}, f \in \mathcal{F}_n, x \in A^n$.

In the sequel, special notations for some classes of poly-algebras will be used. Thus, $Pol(\mathcal{F})$, $Reg(\mathcal{F})$, $Parc(\mathcal{F})$, $Ual(\mathcal{F})$, $\mathcal{E}(\mathcal{F})$, $\mathcal{O}(\mathcal{F})$ is the class of all the poly$-\mathcal{F}-$algebras, regular poly$-\mathcal{F}-$algebras, partial$-\mathcal{F}-$algebras, universal $\mathcal{F}-$algebras, unit poly$-\mathcal{F}-$algebras, and singular poly$-\mathcal{F}-$algebras, respectively. Because of the assumption that the type $\mathcal{F}$ will be fixed in the sequel, we shall omit the symbol $\mathcal{F}$ from these notations. If $\mathcal{K}$ is a class of poly$-\mathcal{F}-$algebras and α a cardinal number, then $\mathcal{K}^+, \mathcal{K}(\alpha), \mathcal{K}[\alpha]$ is the class $\mathcal{K} \cap Reg, \{\mathcal{A} \in \mathcal{F}(A) \cap \mathcal{K} : |A| < \alpha\}, \{\mathcal{A} \in \mathcal{F}(A) \cap \mathcal{K} : |A| \leq \alpha\}$, respectively. A poly-algebra $\mathcal{A} \in \mathcal{F}(A)$ will be called *constant* if there is a $M \subseteq A$ such that for all $n \in \mathbf{N}, f \in \mathcal{F}_n, x \in A^n$ it holds that $f^{\mathcal{A}}(x) = M$. These algebras will be denoted also by (A, M). For example, $o(A) = (A, \emptyset), e(A) = (A, A)$. The class of the constant poly$-\mathcal{F}-$algebras will be denoted by $Con(\mathcal{F})$ (or simply by Con). Instead of "a poly-algebra" we shall often say "an object".

The notions of $\mathcal{F}-$*terms over a set* X is defined in the usual way (see, for example [1], II.1.). Namely, if X is a non-empty set, disjoint with $\mathcal{F}$, then the set $X(\mathcal{F})$ of all $\mathcal{F}-$terms over X is the least set of finite sequences on $X \cup \mathcal{F}$ satisfying the following conditions:

(i) $X \cup \mathcal{F}_0 \subseteq X(\mathcal{F})$

(ii) $f \in \mathcal{F}_n, n \geq 1, t_1, \ldots, t_n \in X(\mathcal{F}) \Longrightarrow f(t_1, t_2, \ldots, t_n) \in X(\mathcal{F})$.

A mapping $h : X \to Y$ can be extended to a mapping $g : X(\mathcal{F}) \to Y(\mathcal{F})$ defined as follows:

(iii) $t = x, x \in X \Rightarrow g(x) = h(x)$, (iv) $f \in \mathcal{F}_0 \Rightarrow g(f) = f$,

(v) $t = f(t_1, \ldots, t_n) \Rightarrow g(t) = f(g(t_1), \ldots, g(t_n))$.

Assume now that $\mathcal{A} \in \mathcal{F}(A), X \subseteq A, t \in X(\mathcal{F})$. Define a mapping $t \to t^{\mathcal{A}}$ from $X(\mathcal{F})$ into $\mathcal{P}(A)$ in the following way:

(vi) $t \in X, t = b \Rightarrow t^{\mathcal{A}} = \{b\}$; (vii) $t \in \mathcal{F}_0, t = f \Rightarrow t^{\mathcal{A}} = f^{\mathcal{A}}$;

(viii) If $t = f(t_1, \ldots, t_n) \in X(\mathcal{F}), f \in \mathcal{F}_n, n \geq 1, t_1, \ldots, t_n \in X(\mathcal{F})$, then $t^{\mathcal{A}} = f^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}})$.

(We remind the reader that:

$$f^{\mathcal{A}}(M_1, \ldots, M_n) = \cup\{f^{\mathcal{A}}(a_1, \ldots, a_n) : a_k \in M_k, 1 \leq k \leq n\}.)$$

We say that $t^{\mathcal{A}}$ is the *value* of t in $\mathcal{A}$.

2. Homomorphisms

Here we shall define three kinds of homomorphisms, every one of which coincides with the usual notion of homomorphism in Ual. Let $\mathcal{A} \in \mathcal{F}(A), \mathcal{B} \in \mathcal{F}(B)$ and $h : A \to B$. The corresponding mapping from $\mathcal{P}(A)$ into $\mathcal{P}(B)$ and from A^n into B^n induced by h will be also denoted by h.

Definition 2.1. *Let $\mathcal{A} \in \mathcal{F}(A), \mathcal{B} \in \mathcal{F}(B)$ and $h : A \to B$. We say that $h : \mathcal{A} \to \mathcal{B}$ is an $i-$*homomorphism *iff for every $f \in \mathcal{F}_n, x \in A^n$ the following condition (H_i) $(i \in \{1,2,3\})$ holds:*

(H_1) $h(f^{\mathcal{A}}(x)) = f^{\mathcal{B}}(h(x))$;

(H_2) $h(f^{\mathcal{A}}(x)) \subseteq f^{\mathcal{B}}(h(x))$;

(H_3) $h(f^{\mathcal{A}}(x)) \supseteq f^{\mathcal{B}}(h(x))$.

The set of all the $i-$homomorphisms $h : \mathcal{A} \to \mathcal{B}$ will be denoted by $\mathcal{H}_i(\mathcal{A}, \mathcal{B})$.

Note that in case $n = 0, f^{\mathcal{A}}(x) = f^{\mathcal{A}}$ and we have $h(f^{\mathcal{A}}) = f^{\mathcal{B}}, h(f^{\mathcal{A}}) \subseteq$

$f^{\mathcal{B}}, h(f^{\mathcal{A}}) \supseteq f^{\mathcal{B}}$, respectively. (We shall usually omit the expression "for every $f \in \mathcal{F}_n, n \geq 0, x \in A^n$".) The following proposition shows that all of this definition of homomorphisms reduce in the special case of universal algebras to the usual definition of homomorphism.

Proposition 2.2. *If $\mathcal{A}, \mathcal{B} \in Ual$ then $\mathcal{H}_1(\mathcal{A},\mathcal{B}) = \mathcal{H}_2(\mathcal{A},\mathcal{B}) = \mathcal{H}_3(\mathcal{A},\mathcal{B}) = \mathcal{H}(\mathcal{A},\mathcal{B})$, where $\mathcal{H}(\mathcal{A},\mathcal{B})$ is the set of all the (usual) homomorphisms $h : \mathcal{A} \to \mathcal{B}$.*

One of the most important properties of homomorphisms is that "we can composed them".

Proposition 2.3. *For every $i \in \{1,2,3\}$, a superposition of $i-$homomorphisms is also an $i-$homomorphism.*

From this proposition we get that Pol is the class of objects of three categories, $Pol(1), Pol(2)$, and $Pol(3)$, where, for each $i \in \{1,2,3\}$, morphisms of $Pol(i)$ are $i-$homomorphisms.

What relationship do we have between these three kinds of homomorphisms? It is easy to see that if $h : \mathcal{A} \to \mathcal{B}$ is a 1–homomorphism, then it is also an $i-$homomorphism for $i = 2$ and $i = 3$, and conversely:

Proposition 2.4. *For any $\mathcal{A}, \mathcal{B} \in Pol$, it holds that $\mathcal{H}_1(\mathcal{A},\mathcal{B}) = \mathcal{H}_2(\mathcal{A},\mathcal{B}) \cap \mathcal{H}_3(\mathcal{A},\mathcal{B})$.*

The following example shows us that there are 2–homomorphisms which are not 3–homomorphisms and conversely:

Example 2.5. *Let A be a non-empty set and $1 : A \to A$ the identity mapping. Then $1 : e(A) \to o(A)$ is a 3–homomorphism but not a 2–homomorphism. Also, $1 : o(A) \to e(A)$ is a 2–homomorphism which is not a 3–homomorphism.*

Proposition 2.6. *Let $\mathcal{A} \in \mathcal{F}(A), \mathcal{B} \in \mathcal{F}(B)$, and h be a mapping from A into B; Denote also by h the corresponding mapping g from $A(\mathcal{F})$ into $B(\mathcal{F})$ induced by h. Then, for every $t \in A(\mathcal{F})$ the following implications hold:*

1) $h \in \mathcal{H}_1(\mathcal{A},\mathcal{B}) \Rightarrow h(t^{\mathcal{A}}) = h(t)^{\mathcal{B}}$,
2) $h \in \mathcal{H}_2(\mathcal{A},\mathcal{B}) \Rightarrow h(t^{\mathcal{A}}) \subseteq h(t)^{\mathcal{B}}$,
3) $h \in \mathcal{H}_3(\mathcal{A},\mathcal{B}) \Rightarrow h(t^{\mathcal{A}}) \supseteq h(t)^{\mathcal{B}}$.

Proposition 2.7. *If $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A} \in \mathcal{F}(A)$ and $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B} \in \mathcal{F}(B)$ are such that $\mathcal{A}_1 \leq \mathcal{A}_2, \mathcal{B}_1 \leq \mathcal{B}_2$ then the following inclusions hold:*

a) $\mathcal{H}_2(\mathcal{A}_2,\mathcal{B}) \subseteq \mathcal{H}_2(\mathcal{A}_1,\mathcal{B})$; b) $\mathcal{H}_3(\mathcal{A}_1,\mathcal{B}) \subseteq \mathcal{H}_3(\mathcal{A}_2,\mathcal{B})$;
c) $\mathcal{H}_2(\mathcal{A},\mathcal{B}_1) \subseteq \mathcal{H}_2(\mathcal{A},\mathcal{B}_2)$; d) $\mathcal{H}_3(\mathcal{A},\mathcal{B}_2) \subseteq \mathcal{H}_3(\mathcal{A},\mathcal{B}_1)$

Proposition 2.8. *If $\mathcal{A} \in \mathcal{F}(A), \mathcal{B} \in \mathcal{F}(B)$, then the following implications hold:*

a) $\mathcal{A} = o(\mathcal{A})$ *or* $\mathcal{B} = e(B) \Rightarrow \mathcal{H}_2(\mathcal{A},\mathcal{B}) = B^A$ *(B^A is the set of all the mappings $h : A \to B$)*;
b) $\mathcal{B} = o(B) \Rightarrow (\mathcal{H}_2(\mathcal{A},\mathcal{B}) \neq \emptyset \Leftrightarrow \mathcal{A} = o(\mathcal{A}))$;
c) $\mathcal{B} = o(B) \Rightarrow \mathcal{H}_3(\mathcal{A},\mathcal{B}) = B^A$;
d) $\mathcal{A} = e(A)$ *and* $h : A \to B$ *is surjective* $\Rightarrow h \in \mathcal{H}_3(\mathcal{A},\mathcal{B})$;
e) $\mathcal{H}_3(\mathcal{A},\mathcal{B}) \neq \emptyset$ *and* $\mathcal{B} \in Reg \Rightarrow \mathcal{A} \in Reg$;
f) *If there is a surjective* $h \in \mathcal{H}_2(\mathcal{A},\mathcal{B})$ *and* $\mathcal{A} \in Reg$ *then* $\mathcal{B} \in Reg$.

It is natural to give the following definition of an $i-$isomorphism:

Definition 2.9. *Let $\mathcal{A} \in \mathcal{F}(A), \mathcal{B} \in \mathcal{F}(B)$, and let $h : A \to B$ be a bijection. Then, we say that $h : \mathcal{A} \to \mathcal{B}$ is an $i-$*isomorphism *iff $h \in \mathcal{H}_i(\mathcal{A},\mathcal{B})$ and $h^{-1} \in \mathcal{H}_i(\mathcal{B},\mathcal{A})$.*

We show below that we have only one kind of $i-$isomorphism for any $i \in \{1,2,3\}$. From Prop. 2.3. we also get that for $i \in \{1,2,3\}$ a superposition of $i-$isomorphisms is also an $i-$isomorphism.

Proposition 2.10. *Let $\mathcal{A}\in\mathcal{F}(A)$, $\mathcal{B}\in\mathcal{F}(B)$, and $h:A\to B$ is a bijection then:*

a) $h \in \mathcal{H}_1(\mathcal{A},\mathcal{B}) \Leftrightarrow h^{-1} \in \mathcal{H}_1(\mathcal{B},\mathcal{A})$;

b) $h \in \mathcal{H}_2(\mathcal{A},\mathcal{B}) \Leftrightarrow h^{-1} \in \mathcal{H}_3(\mathcal{B},\mathcal{A})$.

Proposition 2.11. *Let $\mathcal{A} \in \mathcal{F}(A), \mathcal{B} \in \mathcal{F}(B)$, and $h : A \to B$ is a bijection then the following statements are equivalent:*

a) $h \in \mathcal{H}_1(\mathcal{A},\mathcal{B})$; *b) h is a 1–isomorphism;*

c) h is a 2–isomorphism; *d) h is a 3–isomorphism.*

The last results suggest to say "isomorphism" instead of "i–isomorphism", for every $i \in \{1,2,3\}$. As usualy, we say that $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic* and write $\mathcal{A} \cong \mathcal{B}$ iff there is an isomorphism $h : \mathcal{A} \to \mathcal{B}$.

Remark 2.12. Three kinds of homomorphisms for partial algebras are considered in [5]. Namely, if $\mathcal{A},\mathcal{B} \in Parc$, then "$h \in \mathcal{H}_2(\mathcal{A},\mathcal{B})$"."$h \in \mathcal{H}_1(\mathcal{A},\mathcal{B})$" in [5] means "$h : \mathcal{A} \to \mathcal{B}$ is a homomorphism", "$h : \mathcal{A} \to \mathcal{B}$ is a strong homomorphism", respectively. "Full homomorphisms" are the third kind of homomorphisms considered in [5]. To obtain the notion of "full homomorphisms" in Pol, we have to make a modification of the definition of 3–homomorphisms. Namely, if $\mathcal{A} \in \mathcal{F}(A), \mathcal{B} \in \mathcal{F}(B), h : \mathcal{A} \to \mathcal{B}$, let us say that $h : \mathcal{A} \to \mathcal{B}$ is a *3'–homomorphism* iff the following inclusion $(H_{3'})$ holds:

$$(H_{3'}) \qquad f^{\mathcal{B}}(h(x)) \cap h(A) \subseteq h(f^{\mathcal{A}}(h(x))),$$

for every $f \in \mathcal{F}_n, n \geq 1, x \in A^n$. (If $f \in \mathcal{F}_0$, then $(H_{3'})$ gets the following form: $f^{\mathcal{B}} \cap h(\mathcal{A}) \subseteq h(f^{\mathcal{A}})$.) For $h : \mathcal{A} \to \mathcal{B}$ we say that it is a *full homomorphism* iff h is both a 2–homomorphism and 3'–homomorphism. We also note that in the case of multi-groupoids i.e. poly–2–groupoids, "homomorphism" means the same as our "1–homomorphism". (See, for example, [1], II.7.)

Congruences on poly-algebras can be defined as kernels of homomorphisms. Namely, if $\mathcal{A} \in \mathcal{F}(A)$, then an equivalence $\approx$ in A is called an i*–congruence* on $\mathcal{A}$ iff there is a $h \in \mathcal{H}_i(\mathcal{A},\mathcal{B})$ such that $\approx$ is $\ker(h)$, i.e.

$$(\forall x, y \in A)(x \approx y \Leftrightarrow h(x) = h(y)).$$

The following statements imply that only 1–congruences can be of some interest.

Proposition 2.13. *If $i \in \{2,3\}$, then every equivalence $\approx$ in A is an i–congruence on $\mathcal{A} \in \mathcal{F}(A)$.*

Proof. Denote by $\bar{A}$ the quotient set $A_{/\approx} = \{\bar{x} : x \in A\}$, where $\bar{x}$ is the $\approx$ –equivalence class containing x, i.e. $\bar{x} = \{y \in A : x \approx y\}$. Denote also by $\bar{h}$ the corresponding natural mapping $\bar{h} : x \to \bar{x}$. Then $\bar{h} \in \mathcal{H}_2(\mathcal{A}, e(\bar{A})) \cap \mathcal{H}_3(\mathcal{A}, o(\bar{A}))$, and $\approx$ is $\ker(\bar{h})$. □

Let $\approx$ be an equivalence in A, and $\mathbf{X},\mathbf{Y} \subseteq A$. We shall write $\mathbf{X} \approx \mathbf{Y}$ if it holds that

$$(\forall x \in \mathbf{X})(\forall y \in \mathbf{Y})(\exists u \in \mathbf{Y})(\exists v \in \mathbf{X})(x \approx u \wedge y \approx v).$$

Proposition 2.14. *An equivalence $\approx$ in A is a 1–congruence on $\mathcal{A} \in \mathcal{F}(A)$ iff the following condition holds:*

$$(*) \qquad x \approx y \Rightarrow f^{\mathcal{A}}(x) \approx f^{\mathcal{A}}(y),$$

for every $f \in \mathcal{F}_n, n \geq 1, x, y \in A^n$.

Proof. Assume that $h : \mathcal{A} \to \mathcal{B}$ is a 1–homomorphism, $f \in \mathcal{F}_n$. If $\approx$ is the kernel of h and if $x, y \in A^n$ are such that $x \approx y$, then we have $h(x) = h(y)$, i.e. $h(f^{\mathcal{A}}(x)) = f^{\mathcal{B}}(h(x)) = f^{\mathcal{B}}(h(y)) = h(f^{\mathcal{A}}(y))$, and this implies that $f^{\mathcal{A}}(x) \approx f^{\mathcal{A}}(y)$. Therefore, every 1–congruence $\approx$ satisfies $(*)$. Conversely, let $\approx$ be an equivalence on A such that $(*)$ holds. Then, we can define an algebra $\bar{\mathcal{A}} \in \mathcal{F}(\bar{A})$, by

$$f^{\bar{\mathcal{A}}}(\bar{x}) = \{\bar{u} : u \in f^{\mathcal{A}}(x)\},$$

for every $f \in \mathcal{F}_n, n \geq 0, x \in A^n$. It can be easily seen that $\bar{\mathcal{A}}$ is well defined and $\bar{h} \in \mathcal{H}_1(\mathcal{A}, \bar{\mathcal{A}})$. □

All statements about 1–homomorphism suggest to say "homomorphism" instead of "1–homomorphism", and in the same way "congruence" instead of "1–congruence"

Proposition 2.15. *The intersection of two congruences is not necessarily a congruence.*

Proof. Assume that $\mathcal{F} = \mathcal{F}_1 = \{f\}$. Let $A = \{b, c, b_1, b_2, c_1, c_2\}$ and let $\mathcal{A} \in \mathcal{F}(A)$ be defined by $f(b) = \{b_1, b_2\}, f(c) = \{c_1, c_2\}, f(b_1) = f(b_2) = f(c_1) = f(c_2) = \emptyset$.

If $A/_{\approx_1} = \{\{b, c\}, \{b_1, c_2\}, \{c_1, b_2\}\}$, $A/_{\approx_2} = \{\{b, c\}, \{b_1, c_1\}, \{b_2, c_2\}\}$, then $\approx_1$ and $\approx_2$ are congruences. Their intersection $\approx$ is not a congruence. Namely, we have $b \approx c$ but $f(b) = \{b_1, b_2\} \not\approx \{c_1, c_2\} = f(c)$. □

3. Subalgebras

The notion of a subobject is defined in the usual way.

Definition 3.1. *Let $\mathcal{A} \in \mathcal{F}(A), \mathcal{B} \in \mathcal{F}(B)$ be such that $B \subseteq A$. We say that $\mathcal{B}$ is a* subalgebra *of $\mathcal{A}$ iff $f^{\mathcal{B}}(x) = f^{\mathcal{A}}(x)$ for every $f \in \mathcal{F}_n, n \geq 0, x \in B^n$. The set of all the subalgebras of $\mathcal{A}$ will be denoted by $\mathcal{S}(\mathcal{A})$.*

Remark 3.2. Note, that the Def. 3.1. is not the only "natural" definition of the notion of subalgebra. Namely, let $\mathcal{A} \in \mathcal{F}(A), \mathcal{B} \in \mathcal{F}(B)$ be such that $B \subseteq A$. We say that $\mathcal{B}$ is a *weak subalgebra* of $\mathcal{A}$ iff for every $f \in \mathcal{F}_n, n \geq 0, x \in B^n$ it holds that:
$$f^{\mathcal{A}}(x) \neq \emptyset \Rightarrow f^{\mathcal{A}}(x) \cap f^{\mathcal{B}}(x) \neq \emptyset.$$
It is easy to see that in the class *Parc* the notions of subalgebra and weak subalgebra coincide. Also, it should be noted that generalizing the corresponding notion in *Parc* (see [5], § 13) we obtain two new kinds of subalgebra. Namely, if $\mathcal{A} \in \mathcal{F}(A), \mathcal{B} \in \mathcal{F}(B)$ and $B \subseteq A$, then we say that $\mathcal{B}$ is a *relative subalgebra of $\mathcal{A}$* iff it holds that:
$$x \in B^n \Rightarrow f^{\mathcal{B}}(x) = f^{\mathcal{A}}(x) \cap B;$$
$\mathcal{B}$ is a *weak relative subalgebra of $\mathcal{A}$* iff it holds that:
$$x \in B^n \Rightarrow f^{\mathcal{B}}(x) \subseteq f^{\mathcal{A}}(x) \cap B.$$
Here, we shall consider only subalgebras defined by Def. 3.1.

Proposition 3.3. *Let $\mathcal{A} \in \mathcal{F}(A), \mathcal{B} \in \mathcal{S}(\mathcal{A})$ and $t \in B(\mathcal{F})$. Then $t^{\mathcal{B}} = t^{\mathcal{A}}$.*

Proposition 3.4. *If $\mathcal{A} \in \mathcal{F}(A), \mathcal{B} \in \mathcal{F}(B)$ and $B \subseteq A$, then the following implications hold:*

a) $\mathcal{A} = o(A) \Rightarrow (\mathcal{B} \in \mathcal{S}(\mathcal{A})$ iff $\mathcal{B} = o(B))$;

b) $\mathcal{A} = e(A) \Rightarrow (\mathcal{B} \in \mathcal{S}(\mathcal{A})$ iff $\mathcal{B} = \mathcal{A})$.

Proposition 3.5. *If $\mathcal{A} \in \mathcal{F}(A), \emptyset \neq B \subseteq A$, then there is at most one $\mathcal{B} \in \mathcal{F}(B)$ which is a subalgebra of $\mathcal{A}$. Such a subalgebra $\mathcal{B} \in \mathcal{F}(B)$ exists iff:*

$(*)$ $\quad f^{\mathcal{A}}(x) \subseteq B$, *for every* $f \in \mathcal{F}_n, n \geq 0, x \in B^n$.

Let $\mathcal{A} \in \mathcal{F}(A)$ and $\emptyset \neq B \subseteq A$. We say that B is a *subobject* of $\mathcal{A}$ iff $(*)$ holds.

Proposition 3.6. *If B is a subobject of $\mathcal{A}$, then $t^{\mathcal{A}} \subseteq B$, for every $t \in B(\mathcal{F})$.*

The following statement implies the notion of "subobject generated by a subset".

Proposition 3.7. *A non-empty intersection of subobjects is a subobject.*

If $\mathcal{A} \in \mathcal{F}(A)$, and $X \subseteq A, X \neq \emptyset$, then by Prop 3.7., there exists the least subobject B of $\mathcal{A}$ such that $X \subseteq B$. We shall write $B = \langle X \rangle_{\mathcal{A}}$, and say that $\langle X \rangle_{\mathcal{A}}$ is *generated by* X. If there exists a least subobject B of $\mathcal{A} \in \mathcal{F}(A)$ then we shall write $B = \langle \emptyset \rangle_{\mathcal{A}}$; othervise, $\langle \emptyset \rangle_{\mathcal{A}}$ is meaningless. Namely, $\langle \emptyset \rangle_{\mathcal{A}}$ exists if there is a $c \in A$ such that $c \in \cap\{\langle x \rangle_{\mathcal{A}} : x \in A\}$, and then $\langle \emptyset \rangle_{\mathcal{A}} = \langle c \rangle_{\mathcal{A}}$.

The notion of $\mathcal{F}-$terms over set a X can be used for giving a description of the subobject $\langle X \rangle_{\mathcal{A}}$ of $\mathcal{A}$ generated by X.

Proposition 3.8. *If $\mathcal{A}\in\mathcal{F}(A)$ and $\emptyset\neq B\subseteq A$ then $\langle B\rangle_{\mathcal{A}}=\cup\{t^{\mathcal{A}}:t\in B(\mathcal{F})\}$.*

Remark 3.9. We can define the algebra $B(\mathcal{F})$ also in the case $B=\emptyset$. The definition of the algebra $\emptyset(\mathcal{F})$ of $\mathcal{F}-$terms over $\emptyset$ is the same as in the case $B\neq\emptyset$. But, the Proposition 3.8. does not hold in the case $B=\emptyset$. For example, if $\mathcal{F}_0=\emptyset$, then for all $\mathcal{A}\in\mathcal{F}(A)$, we have $\{t^{\mathcal{A}}:t\in\emptyset(\mathcal{F})\}=\emptyset$. On the other hand, if we assume that $\mathcal{F}\neq\mathcal{F}_0$ then for $\mathcal{A}=e(A)$ we have $\langle\emptyset\rangle_{\mathcal{A}}=A\neq\cup\{t^{\mathcal{A}}:t\in B(\mathcal{F})\}=\emptyset$. Even in the case $\mathcal{F}_0\neq\emptyset$ (and $B=\emptyset$) the equation from Proposition 3.7. can be not true. For example, let $\mathcal{A}$ be a poly-algebra such that for all $f\in\mathcal{F}_0(\mathcal{F}_0\neq\emptyset)$ it holds that $f^{\mathcal{A}}=\emptyset$, and for all $f\in\mathcal{F}_n(n\geq 1)$, and all $x\in A^n, f^{\mathcal{A}}(x)=A$. In this case we also have $\cup\{t^{\mathcal{A}}:t\in B(\mathcal{F})\}=\emptyset$, but $\langle\emptyset\rangle_{\mathcal{A}}=A$.

Proposition 3.10. *Let $\mathcal{A}\in\mathcal{F}(A)$ and B be a non-empty subset of A. Then the following statements are equivalent:*

a) B is a subobject of $\mathcal{A}$;

b) There is a unique object $\mathcal{B}\in\mathcal{F}(B)$ such that the embedding from B into A is a 1-homomorphism from $\mathcal{B}$ into $\mathcal{A}$.

Proposition 3.11. *If $\mathcal{A}\in\mathcal{F}(A),\mathcal{A}'\in\mathcal{F}(A')$ and $h:\mathcal{A}\to\mathcal{A}'$ is an injective 1-homomorphism, then $B=h(A)$ is a subobject of $\mathcal{A}'$ such that the corresponding subalgebra $\mathcal{B}\in\mathcal{F}(B)$ is isomorphic to $\mathcal{A}$.*

Proposition 3.12

1) A 1-homomorphic image of a subobject is also a subobject, and also, a non-empty inverse homomorphic image of a subobject is a subobject.

2) A 3-homomorphic image of a subobject is also a subobject.

3) A non-empty inverse 2-homomorphic image of a subobject is also a subobject.

We can see that only 1–homomorphisms have the "preserving property" in both directions.

Example 3.13. *Let $|A|\geq 2$ and B be a non-empty proper subset of A. Then $1:o(A)\to e(A)$ is a 2-homomorphism, B is a subobject of $o(A)$, but $1(B)=B$ is not a subobject of $e(A)$. Conversely, $1:e(A)\to o(A)$ is a 3-homomorphism, $B=1^{-1}(B)$ is an inverse 3-homomorphic image of the subobject B of $o(A)$, which is not a subobject of $e(A)$.*

4. Free objects

Let $\mathcal{A}\in\mathcal{F}(A)$, $\mathcal{C}\in\mathcal{F}(C)$ and $B\subseteq A$. We say that B is an *$i-$basis of $\mathcal{A}$ over $\mathcal{C}$* if the following two conditions hold:

1) B generates $\mathcal{A}$;
2) For every mapping $h:B\to C$ there is an extension $g:A\to C$ which is an $i-$homomorphism $g:\mathcal{A}\to\mathcal{C}$.

If $\mathcal{K}$ is a class of $\mathcal{F}-$poly-algebras then B is an *$i-$basis of $\mathcal{A}$ over $\mathcal{K}$* iff B is an $i-$basis of $\mathcal{A}$ over every object $\mathcal{C}\in\mathcal{K}$. In this case, if $\mathcal{A}\in\mathcal{K}$, we say that B is an *$i-$basis of $\mathcal{A}$ in $\mathcal{K}$*.

$\mathcal{A}$ is said to be *$i-$free over $\mathcal{C}$* iff there is an $i-$basis B of $\mathcal{A}$ over $\mathcal{C}$. If there is an $i-$basis B of $\mathcal{A}$ over every object $\mathcal{C}\in\mathcal{K}$ then we say that $\mathcal{A}$ is *$i-$free over $\mathcal{K}$*; if, moreover $\mathcal{A}\in\mathcal{K}$, then is an *$i-$free object in $\mathcal{K}$*.

It is well known that if $\mathcal{A}\in\mathcal{F}(A),\mathcal{C}\in\mathcal{F}(C)$ are universal algebras and B is a basis of $\mathcal{A}$ over $\mathcal{C}$, then for every $h:B\to C$ there exists a unique homomorphism $g:\mathcal{A}\to\mathcal{C}$ which is an extension of h. In most of the following examples for poly-algebras, the set of such homomorphisms is infinite. It is also well known that any two free members with the same bases, in a class of universal algebras, are isomorphic. In the case of poly-algebras we have many examples of non-isomorphic $i-$free algebras with the same $i-$bases.

Proposition 4.1. *Let $\mathcal{A}, \mathcal{C} \in Ual$. The following statements are equivalent:*

a) B is a 1–basis of $\mathcal{A}$ over $\mathcal{C}$; *b) B is a 2–basis of $\mathcal{A}$ over $\mathcal{C}$;*

c) B is a 3–basis of $\mathcal{A}$ over $\mathcal{C}$; *d) B is a basis of $\mathcal{A}$ over $\mathcal{C}$.*

Proposition 4.2. *If $\mathcal{K} \subseteq \mathcal{L}$ are two classes and if an object $\mathcal{A}$ is i–free over $\mathcal{L}$, then $\mathcal{A}$ is also i–free over $\mathcal{K}$.*

Proposition 4.3. *Let $\mathcal{A} \in \mathcal{F}(A)$ be 2–free over $\mathcal{D} \in \mathcal{F}(D)$ with an 2–basis B. Let $\mathcal{H}$ be a subobject of $\mathcal{D}$. Then $\mathcal{A}$ is 2–free over $\mathcal{H}$, with the same 2–basis B.*

Proposition 4.4. *If B is a 1–basis of $\mathcal{A}$ over $\mathcal{C}$, then B is both a 2–basis and a 3–basis of $\mathcal{A}$ over $\mathcal{C}$.*

This proposition suggests to say "basis" and "free" instead of "1–basis" and "1–free", respectively.

In the following we shall investigate i–free objects over (and in) some classes of poly-algebras.

Proposition 4.5. *A singular object $o(A)$ is 2–free over any object $\mathcal{B}$. If $|A| \geq 2$ then A is the unique 2–basis of $o(A)$ over $\mathcal{B}$. If $|A| = 1$, then $\emptyset$ and A are a 2–basis of $o(A)$ over $\mathcal{B}$.*

Proposition 4.6. *Let $\mathcal{A}$ be a 2–free object with 2–basis B over a class $\mathcal{K}$, and let $\mathcal{D} \in \mathcal{F}(D)$ be a non-regular object in $\mathcal{K}$. If one of the following conditions hold: a) $|B| \geq |D|$; or b) B is infinite; then $\mathcal{A}$ is also non-regular.*

Proof. From the non-regularity of $\mathcal{D}$ it follows that $f^{\mathcal{D}}(d) = \emptyset$, for some $f \in \mathcal{F}_n, d \in D^n, n \geq 0$. Either of the conditions a), b) implies that there exists an $h : B \to D$ and $b \in B^n$ such that $h(b) = d$. If $g : \mathcal{A} \to \mathcal{D}$ is a 2–homomorphism which is an extension of h, then we have $g(f^{\mathcal{A}}(b)) \subseteq f^{\mathcal{D}}(g(b)) = f^{\mathcal{D}}(d) = \emptyset$, and this implies $f^{\mathcal{A}}(b) = \emptyset$. □

Proposition 4.7. *Let $\mathcal{B}$ be an object such that there exists a singular subobject of $\mathcal{B}$. Then $\mathcal{A} \in \mathcal{F}(A)$ is 2–free over $\mathcal{B}$ iff $\mathcal{A} = o(A)$.*

Proof. Let D be a singular subobject of $\mathcal{B}$. Then $\mathcal{A}$ is 2–free over $\mathcal{D} = o(D)$, as well, and therefore there exists a 2–homomorphism $h : \mathcal{A} \to o(D)$, which implies that $\mathcal{A} = o(A)$. □

As a corollary we obtain the following:

Proposition 4.8. *Let $\mathcal{K} \in \{Pol,\ Con,\ Pol(\alpha),\ Con(\alpha),\ Pol[\beta],\ Con[\beta]\}$, where α, β are cardinals such that $\alpha > 0$. Then, an object $\mathcal{A} \in \mathcal{F}(A)$ is 2–free over (or in) $\mathcal{K}$ iff $\mathcal{A} = o(A)$.*

Here we consider some subclasses of Reg.

Proposition 4.9. *Every generating set of an object $\mathcal{A}$ is a 2–basis of $\mathcal{A}$ over $\mathcal{E}$. Specially, if $\mathcal{A}=e(A)$, then any subset B of A is a 2–basis of $\mathcal{A}$ in $\mathcal{E}$.*

Proposition 4.10. *Every object $(A, M) \in Con^+$ is 2–free in Con^+. B is a 2–basis of (A, M) in Con^+ iff $B = A \backslash M$.*

Below we state some results concerning i–free objects, for $i \in \{1, 3\}$.

Proposition 4.11. *Every generating subset of an object $\mathcal{A}$ is a 3–basis of $\mathcal{A}$ over $\mathcal{O}$. Every object $o(A) \in \mathcal{O}$ is free in $\mathcal{O}$.*

Proposition 4.12. *If there is a non-regular 3–free object $\mathcal{A}$ over a class $\mathcal{K}$, then every object of $\mathcal{K}$ is also non-regular.*

Proposition 4.13. *Let $\mathcal{A} \in \mathcal{F}(A)$ be a free object over $\mathcal{K}$ with a non-empty basis B, and assume that there is at least one regular object in $\mathcal{K}$.*

a) If B is infinite then every object in $\mathcal{K}$ is regular.

b) If B is finite and if $\mathcal{D} \in \mathcal{F}(D) \cap \mathcal{K}$ is such that $|B| \geq |D|$, then $\mathcal{D}$ is regular.

Proof. Proposition 4.12. implies that $\mathcal{A}$ is regular. Either of the assumptions a), b) implies that for every $f \in \mathcal{F}_n, d \in D^n, n \geq 0$, there is an $h : B \to D$ and a $b \in B^n$ such that $h(b) = d$, and then we should apply the assumption that B is a basis of $\mathcal{A}$ over $\mathcal{K}$, and the fact that $\mathcal{A}$ is regular. □

Proposition 4.14. *Let α be an infinite cardinal, $\mathcal{K} = \{e(A) : |A| \leq \alpha\}$, and $\mathcal{L} = \mathcal{O} \cup \mathcal{K}$. Then:*

a) $e(D)$ is a free object in $\mathcal{K}$ iff $|D| = \alpha$, and $B \subseteq D$ is a basis of $e(D)$ in $\mathcal{K}$ iff $|D \backslash B| = \alpha$.

b) Every object $e(A) \in \mathcal{K}$ is 2–free in $\mathcal{K}$ and any subset B of A is a 2–basis of $e(A)$ in $\mathcal{K}$.

c) $\mathcal{A} \in \mathcal{L}$ is a 3–free object in $\mathcal{L}$ iff $\mathcal{A}$ is a free object in $\mathcal{K}$.

d) There is no free object in $\mathcal{L}$.

Proposition 4.15. *If $\mathcal{K} \in \{\mathcal{E}, Con, Pol, Reg, Con^+\}$ then there is no 3–free object over $\mathcal{K}$.*

Proof. $\mathcal{E}$ is the least member of $\mathcal{K}$, and thus it is enough to consider the case $\mathcal{K} = \mathcal{E}$. If $\mathcal{A} \in \mathcal{F}(A), e(M) \in \mathcal{E}$ are such that $|A| < |M|$, then there is no 3–homomorphism $h : \mathcal{A} \to e(M)$. □

As a corollary of Propositions 4.13., 4.14., 4.15. we obtain the following:

Proposition 4.16. *If $\mathcal{K} \in \{\mathcal{E},\ Con,\ Pol,\ Reg,\ Pol(\alpha),\ Pol[\beta],\ Con(\alpha),\ Con[\beta], Con^+\}$, where $\alpha > 0$, then there is no free object over $\mathcal{K}$.*

In all the examples of classes $\mathcal{K}$ containing 3–free objects obtained above we had the situation that $\mathcal{K}$ contained a 2–free object as well. Now we shall give a class with 3–free objects without 2–free objects.

Proposition 4.17. *Let $\mathcal{K}$ be the class of all the non-singular objects $\mathcal{A} \in \mathcal{F}(A)$, such that $|A| \leq \alpha$, where α is an infinite cardinal. Then there is no 2–free object in $\mathcal{K}$. $\mathcal{A} \in \mathcal{F}(A)$ is a 3–free object in $\mathcal{K}$ iff $\mathcal{A} = e(A)$ and $|A| = \alpha$; In this case B is a 3–basis of $e(A)$ in $\mathcal{K}$ iff $|A \backslash B| = \alpha$.*

Proof. Clearly, there exist members of $\mathcal{K}$ with singular subobjects. Thus, by Proposition 4.7., $\mathcal{K}$ does not contain a 2–free object. □

Proposition 4.4. sugests the following

Problem 4.18. Find a pair of objects $\mathcal{A}, \mathcal{B}$ with the following property:

a) $\mathcal{A}$ is not free over $\mathcal{B}$, but $\mathcal{A}$ is both a 2–free and a 3–free over $\mathcal{B}$.

b) There is a $B \subseteq A$ which is both a 2–basis and a 3–basis of $\mathcal{A}$ over $\mathcal{B}$, but B is not a basis of $\mathcal{A}$ over $\mathcal{B}$.

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REZIME

O POLI-ALGEBRAMA

U ovom radu se razmatraju poli-algebre proizvoljnog tipa $\mathcal{F}$. Ak je $\mathcal{F}$ neprazna, disjunktna unija funkcijskih simbola, $\mathcal{F} = \cup\{\mathcal{F}_n : n \geq 0\}$, onda *poli-algebra tipa* $\mathcal{F}$ sa nosačem $A \neq \emptyset$ jeste svako preslikavanje $\mathcal{A} : f \to f^{\mathcal{A}}$ tako da je $f^{\mathcal{A}} : A^n \to \mathcal{P}(A)$, za sve $f \in \mathcal{F}_n, n \geq 0$. Podalgebre se definišu na uobičajen način. U radu se posmatraju tri tipa homomorfizma, koji impliciraju tri tipa slobodnih objekata. Dokazani su rezultati o podalgebrama, homomorfizmima i slobodnim objektima za razne klase poli-algebri.