FREE OBJECTS IN THE CLASS OF VECTOR VALUED GROUPOIDS INDUCED BY SEMIGROUPS

Contributions to General Algebra, Wien, 7 (1991), 87–95

Ǵ. Čupona and S. Markovski

Abstract. The class $Sem(n,m)$ of vector valued groupoids (v. v. g.) induced by semigroups is defined in [3], and there some properties concerning subgroupoids and homomorphisms are established. Here we consider two kind of free objects in the class $Sem(n,m)$: (1) free objects, which have "usual" properties, and (2) weakly free objects which behave "unusual". (For example, it is shown that there are nonisomorphic weakly free objects with same "weak" basis.)

0. Introduction

Algebraic structures with vector valued operations (more specially: (n,m)-operations) are treated by several authors, and a review of papers on this topic can be found in [2]. Every structure with an (n,m)-operation, i. e. every (n,m)-groupoid, is essentially equivalent with a universal algebra (its component algebra) with m n-ary operations. This connection implies that each "universal algebraic notion" can be translated to corresponding "(n,m)-groupoid notion". But, vector valued algebraic structures can be generalized in such a way that this strong connection with universal algebras to be losed. Fully commutative (n,m)-groupoids (quasigroups, semigroups and groups) were the first kind of such generalized vector valued structures ([5], [6]). All known classes of different kinds of (n,m)-groupoids are subclasses of $Sem(n,m)$. The main purpose of this paper are free objects in $Sem(n,m)$.

We proceed by stating some necessary preliminary definitions.

Let $\mathbf{S}=(S,\cdot)$ be a semigroup with a generating subset Q, i. e. $S=\bigcup\{Q_\alpha \mid \alpha\geq 1\}$, where $Q_\alpha=\{a_1\cdot a_2\cdot\ldots\cdot a_\alpha \mid a_i\in Q\}$. If n and m are positive integers and f a mapping from Q_n into Q_m, then we say that $(Q;f)$ is an *$(\mathbf{S};n,m)$-groupoid*, or a *semigroup-(n,m)-groupoid*. More precisely, the ordered triple $\mathbf{Q}=(\mathbf{S};Q;f)$ is a semigroup-(n,m)-groupoid. (We say that Q is the *carrier* and f is the *operation* of $\mathbf{Q}$.) The class of semigroup-(n,m)-groupoids will be denoted by $Sem(n,m)$, and the members of this class will be called simply *objects*.

Further on, we assume that n and m are given positive integers.

Let $\mathbf{Q}=(\mathbf{S};Q;f)$ and $\mathbf{Q}'=(\mathbf{S}';Q';f')$ be two objects. A mapping $\varphi: Q\to Q'$ is said to be a *homomorphism* from $\mathbf{Q}$ into $\mathbf{Q}'$ if for arbitrary $a_\nu, b_\lambda\in Q$ the following implication holds:

$$f(a_1\cdot\ldots\cdot a_n)=b_1\cdot\ldots\cdot b_m \Rightarrow f'(\varphi(a_1)\cdot\ldots\cdot\varphi(a_n))=\varphi(b_1)\cdot\ldots\cdot\varphi(b_m) \qquad (0.1)$$

A homomorphism φ is called an *isomorphism* if it is bijective and $\varphi^{-1}: Q'\to Q$ is a homomorphism too.

Let $\mathbf{Q}=(\mathbf{S};Q;f)$ be an object and P a nonempty subset of Q. Then, for every $\alpha\geq 1$ we have a subset P_α of S defined by $P_\alpha=\{a_1\cdot\ldots\cdot a_\alpha \mid a_i\in P\}$. P is called a *subobject* of $\mathbf{Q}$ iff $f(P_n)\subseteq P_m$, i. e. if $\mathbf{Q}$ induces an object $\mathbf{P}=(\mathbf{T};P;f')$, where $\mathbf{T}$ is the subsemigroup of $\mathbf{S}$ generated by P, and f' is the restriction of f on P. P is called a *strong subobject* of $\mathbf{Q}$ if, for every $a\in P_n$, $b_1,\ldots,b_m\in Q$, the following implication holds:

$$f(a)=b_1\cdot\ldots\cdot b_m \Rightarrow b_1,\ldots,b_m\in P. \qquad (0.2)$$

We state some results which are either trivial or proved in [3].

(i) A bijective homomorphism is not necessarily an isomorphism.

(ii) A strong subobject is a subobject too, but a subobject is not necessarily a strong one.

(iii) A nonempty intersection of strong subobjects is a strong subobject too, but a nonempty intersection of subobjects is not necessarily a subobject.

(iv) A homomorphic image of a subobject is a subobject too, but a nonempty complete inverse homomorphic image of a subobject is not necessarily a subobject.

(iv') A nonempty complete inverse homomorphic image of a strong subobject is a strong subobject, but a homomorphic image of a strong subobject is not necessarily a strong subobject.

(v) If $\mathbf{Q}, \mathbf{Q}' \in Sem(n,m)$ and $\psi : \mathbf{Q} \to \mathbf{Q}'$ is a homomorphism and P is a subobject of $\mathbf{Q}$, then $\psi^* : P \to Q'$ is a homomorphism as well, where $\psi^* = \psi|P$ is the restriction of ψ on P.

In what follows we denote by Q^+ the semigroup freely generated by a nonempty set Q, i. e. Q^+ is the set of nonempty words on an alphabet Q. Then $Gr(n,m)$ denotes the class of all objects $\mathbf{Q} = (\mathbf{S}; Q; f) \in Sem(n,m)$ such that $\mathbf{S} = Q^+$, and then $\mathbf{Q}$ is usually said to be an (n,m)-groupoid ([2]).

(vi) If $\mathbf{Q} \in Gr(n,m)$, then every subobject of $\mathbf{Q}$ is a strong subobject as well, and every bijective homomorphism is an isomorphism.

1. Free objects

Let $\mathbf{Q} = (\mathbf{S}; Q; f) \in Sem(n,m)$.

A nonempty subset B of Q is said to be *free in* $\mathbf{Q}$ iff for every object $\mathbf{Q}' = (\mathbf{S}'; Q'; f') \in Sem(n,m)$ and every mapping $\varphi : B \to Q'$ there is a homomorphism $\psi : \mathbf{Q} \to \mathbf{Q}'$ such that $(\forall b \in B)\ \psi(b) = \varphi(b)$, i. e. ψ is an extension of φ.

A subset C of Q is called a *generating subset* of $\mathbf{Q}$ iff there does not exist a proper subgroupoid P of $\mathbf{Q}$ such that $C \subseteq P$. (According to **0**.(iii) we can not state that "every subset of Q generates a subgroupoid of $\mathbf{Q}$".)

We say that $\mathbf{Q}$ is a *free object* in $Sem(n,m)$ iff there is a generating subset B of Q which is free in $\mathbf{Q}$. (In this case we say that B is a *basis* of $\mathbf{Q}$.)

The next theorem gives a complete description of free objects.

Theorem 1. (i) *Let B be a nonempty set and denote by $([B]; f_1, f_2, \ldots, f_m)$ the (absolutely) free algebra with basis B and m n-ary operations $f_1, f_2, \ldots, f_m$. Define an object* $\mathbf{F(B)} = ([B]^+; [B]; f)$ *by:*

$$f(u_1 \cdot \ldots \cdot u_n) = v_1 \cdot \ldots \cdot v_m \quad \Leftrightarrow \quad v_1 = f_1(u_1, \ldots, u_n), \ldots, v_m = f_m(u_1, \ldots, u_n). \qquad (1.1)$$

Then $\mathbf{F(B)}$ *is a free object in $Sem(n,m)$ with a basis B.*

(ii) *Two free objects in $Sem(n,m)$ with a same basis are isomorphic.*

(iii) *The basis in a free object in $Sem(n,m)$ is uniquely determined.*

(iv) *Let $m \geq 2$. There exists an object*[1] $\mathbf{Q}' = (\mathbf{S}'; Q'; f')$ *such that any mapping $\varphi : B \to Q'$ can be extended to infinitely many homomorphisms*[2] $\psi : \mathbf{F(B)} \to \mathbf{Q}'$.

(v) *Every subobject of a free object in $Sem(n,m)$ is a strong subobject as well.*

Proof. (i) We note that $\mathbf{F(B)}$ is a free (n,m)-groupoid with a basis B in the class $Gr(n,m)$ of (n,m)-groupoids ([2]). Following [1], we can take that $[B]$ is the set of words on B of type $\{f_1, f_2, \ldots, f_m\}$. But, we prefer to give an explicite construction of the set $[B]$, as follows.

Define a collection of sets $\{B_{(\alpha)} | \alpha \geq 0\}$ by

$$B_{(0)} = B, \qquad B_{(\alpha+1)} = B_{(\alpha)} \cup \mathbb{N}_m \times B^n_{(\alpha)}, \qquad (1.2)$$

where $\mathbb{N}_m = \{1, 2, \ldots, m\}$, and $B^n_{(\alpha)}$ is the n-th cartezian power of $B_{(\alpha)}$. Put

$$[B] = \bigcup \{B_{(\alpha)} | \alpha \geq 0\} \qquad (1.3)$$

and define a sequence of n-ary operations $f_1, f_2, \ldots, f_m$ on $[B]$ by:

$$f_i(x) = (i, x) \qquad (1.4)$$

for every $i \in \mathbb{N}_m$, $x \in [B]^n$.

Then $([B]; f_1, f_2, \ldots, f_m)$ is an absolutely free algebra with a basis B and with m n-ary operations, and the (n,m)-groupoid $\mathbf{[B]} = ([B]; f)$ defined by

$$(\forall x \in [B]^n)\ f(x) = (1, x) \cdot \ldots \cdot (m, x) \ \ (= f_1(x) \cdot \ldots \cdot f_m(x)) \qquad (1.1')$$

is a free (n,m)-groupoid with a basis B. (Note that $\mathbf{[B]} = ([B]^+; [B]; f) = \mathbf{F(B)}$.)

We will show that $\mathbf{F(B)}$ is a free object in $Sem(n,m)$, as well.

Let $\mathbf{Q}' = (\mathbf{S}'; Q'; f') \in Sem(n,m)$, and let $\varphi : B \to Q'$ be an arbitrary mapping. We have

[1] Certainly, the class of objects with such a property is infinite.

[2] Thus, in the case $m \geq 2$, it is not true that "if B is a generating subset of $\mathbf{Q} \in Sem(n,m)$, and ψ_1, ψ_2 are two homomorphisms from $\mathbf{Q}$ into an object $\mathbf{Q}' \in Sem(n,m)$ such that $(\forall b \in B)\ \psi_1(b) = \psi_2(b)$, then $\psi_1 = \psi_2$"

to show the existence of a homomorphism $\psi : \mathbf{F(B)} \to \mathbf{Q'}$ which extends φ.

Put $\psi_0 = \varphi$ and assume that for each α, such that $\alpha \leq \gamma$, a mapping $\psi_\alpha : B_{(\alpha)} \to Q'$ is defined with these properties:

1) $\psi_{\delta+1}$ extends ψ_δ, for each $\delta < \gamma$;

2) $f'(\psi_\delta(u_1) \cdot \ldots \cdot \psi_\delta(u_n)) = \psi_{\delta+1}(1,x) \cdot \ldots \cdot \psi_{\delta+1}(m,x)$, for each $\delta < \gamma$ and any $u_1, \ldots, u_n \in B_{(\delta)}$, where $x = u_1 \cdot \ldots \cdot u_n$.

Let $u \in B_{(\gamma+1)}$. If $u \in B_{(\gamma)}$, then we put $\psi_{\gamma+1}(u) = \psi_\gamma(u)$. If $u \in B_{(\gamma+1)} \setminus B_{(\gamma)}$, then there are uniquely determined $i \in \mathbb{N}_m$ and $x = u_1 \cdot \ldots \cdot u_n \in B^n_{(\gamma)}$, such that $u = (i,x)$. By the above assumptions, there are uniquely determined elements $a'_\nu \in Q'$ such that $\psi_\gamma(u_\nu) = a'_\nu$, for every $\nu \in \mathbb{N}_n$ $(= \{1,2,\ldots,n\})$. Then $a' = a'_1 \cdot \ldots \cdot a'_n \in Q'_n$, and therefore $f'(a') = c' \in Q'_m$. This implies that there exists a sequence $c'_1, \ldots, c'_m$ of elements of Q' (not necessarily unique one), such that $c' = c'_1 \cdot \ldots \cdot c'_m$. We choose such a sequence and put $\psi_{\gamma+1}(j,x) = c'_j$ for $j \in \mathbb{N}_m$. $\psi_{\gamma+1}$ is well defined since if $(i,x) \in B_{(\gamma+1)}$ for some $i \in \mathbb{N}_m$, then $(i,x) \in B_{(\gamma+1)}$ for every $i \in \mathbb{N}_m$. By the definition of $\psi_{\gamma+1}$ it follows that the properties 1) and 2) hold also for each $\delta \leq \gamma$.

Denote by ψ the (unique) extension of the chain of mappings $(\psi_\alpha \mid \alpha \geq 0)$. By 1) and 2) we have that $\psi : [B] \to Q'$ is a homomorphism which extends φ. Thus, B is a basis of $\mathbf{F(B)}$.

(ii) Let $\mathbf{Q} = (\mathbf{S}; Q; g)$ be a free object in $Sem(n,m)$ with a basis B. There exist homomorphisms $\xi : \mathbf{F(B)} \to \mathbf{Q}$, $\eta : \mathbf{Q} \to \mathbf{F(B)}$ such that $(\forall b \in B)\ \xi(b) = \eta(b) = b$, and thus $\zeta = \eta\xi$ is an endomorphism of $\mathbf{F(B)}$ such that $(\forall b \in B)\ \zeta(b) = b$. Since the identity automorphism of $\mathbf{F(B)}$ is the unique such an endomorphism of $\mathbf{F(B)}$, it follows that ζ is the identity automorphism of $\mathbf{F(B)}$. Thus $\xi : \mathbf{F(B)} \to \mathbf{Q}$ is an injective mapping, and by **O**.(iv) we have that $\xi(\mathbf{F(B)})$ is a subobject of $\mathbf{Q}$ such that $B \subseteq \xi(F(B))$, which implies that ξ is bijective. Thus $\eta = \xi^{-1}$, and therefore ξ is an isomorphism.

(iii) Clearly B is the unique basis of $\mathbf{F(B)}$, and this, by (ii), implies that the basis of any free object in $Sem(n,m)$ is uniquely determined.

(iv) Let Sl be the variety of semilattices, i.e. commutative and idempotent semigroups. If B is a nonempty set, then we denote by $Sl(B)$ the free semigroup in Sl with basis B. We can assume that $Sl(B)$ is the family of nonempty finite subsets of B, where the operation is the usual set theoretical union. If $\{b_1, \ldots, b_k\} \in Sl(B)$, then we write $\{b_1, \ldots, b_k\} = b_1 \cdot \ldots \cdot b_k$. Denote by $Sl_k(B)$ the set $\{b_1 \cdot \ldots \cdot b_p \mid b_\nu \in B,\ p \leq k\}$. Define a sequence $B_{(0)}, B_{(1)}, \ldots$ by $B_{(0)} = B$, $B_{(\alpha+1)} = B_{(\alpha)} \cup \mathbb{N}_m \times Sl_n(B_{(\alpha)})$, and let $C = \bigcup \{B_{(\alpha)} \mid \alpha \geq 0\}$. An object $\mathbf{C} = (Sl(C); C; g) \in Sem(n,m)$ can be defined by

$$g(x) = (1,x) \cdot \ldots \cdot (m,x).$$

It can be easily seen that there exist infinitely many homomorphisms $\psi : \mathbf{F(B)} \to \mathbf{C}$ such that $(\forall b \in B)\ \psi(b) = b$.

(v) By **O**.(vi), every subobject P of $\mathbf{F(B)}$ is a strong subobject too. By (ii), the same holds in every free object with a basis B. ☐

2. Weakly free objects

Replacing "subgroupoids" by "strong subgroupoids" we obtain a corresponding notion of "weakly free objects" in $Sem(n,m)$.

Namely, if $\mathbf{Q} = (\mathbf{S}; Q; f) \in Sem(n,m)$, and if A is a nonempty subset of Q, then A is called *weakly generating* subset of $\mathbf{Q}$ iff Q is the unique strong subgroupoid of $\mathbf{Q}$ such that $A \subseteq Q$.

An object $\mathbf{Q} \in Sem(n,m)$ is said to be *weakly free* iff there is a weakly generating subset B of Q which is free in $\mathbf{Q}$. Then we also say that B is a *weak basis* of $\mathbf{Q}$.

The following two statements are immediate consequences from the given definitions.

Proposition 2.1. *Every free object with a basis B is a weakly free object with a weak basis B.* ☐

Proposition 2.2. *An object $\mathbf{Q} \in Sem(n,1)$ is weakly free iff it is a free one.* ☐

The main purpose of this section is to obtain a class of weakly free objects in $Sem(n,m)$ which are not free ones.

First, we prove the following

Proposition 2.3. *If* $\mathbf{Q} \in Sem(n,m)$ *is a weakly free object with a weak basis* B*, then there is a subobject* P *of* $\mathbf{Q}$ *such that the corresponding object* $\mathbf{P}$ *is a free object with a basis* B.

Proof. Let $\mathbf{Q}$ be a weakly free object with a weak basis B, and let $\mathbf{F(B)}$ be the free object with a basis B defined in the proof of part (i) of Theorem 1. Thus, there exist homomorphisms $\xi : \mathbf{F(B)} \to \mathbf{Q}$, $\eta : \mathbf{Q} \to \mathbf{F(B)}$, such that $(\forall b \in B)\ \xi(b) = \eta(b) = b$. Then $\eta\xi$ is an endomorphism of $\mathbf{F(B)}$ such that $(\forall b \in B)\ \eta\xi(b) = b$, and therefore $\eta\xi$ is the identity automorphism of $\mathbf{F(B)}$. This implies that ξ is injective, and η is surjective. By **O**.(iv), $P = \xi(F(B))$ $(= \xi([B]))$ is a subobject of $\mathbf{Q}$ and $B \subseteq P$, and the corresponding restriction $\xi^* : \mathbf{F(B)} \to \mathbf{P}$ of ξ is a bijective homomorphism; the restriction $\eta^* : \mathbf{P} \to \mathbf{F(B)}$ of η is also a homomorphism (by **O**.(v)). Moreover, the equality $\eta\xi = 1$ (= the identity automorphism of $\mathbf{F(B)}$) implies that η^* is surjective; thus wa have $\eta^*\xi^* = 1$, i. e. $\xi^{*-1} = \eta^*$ is a homomorphism from $\mathbf{P}$ in $\mathbf{F(B)}$. Thus, $\mathbf{F(B)}$ and $\mathbf{P}$ are isomorphic. □

Now, we will describe a relatively large class of weakly free objects with a given basis.

Let B and Λ be given nonempty sets and n, m positive integers such that $m \geq 2$.

Define a sequence $(B_{(\alpha)} | \alpha \geq 0)$ of sets and a sequence $(\mathbf{S}_{(\alpha)} | \alpha \geq 0)$ of semigroups such that $B_{(\alpha)}$ is a generating subset of $\mathbf{S}_{(\alpha)}$, as follows:

$$B_{(0)} = B, \qquad B_{(1)} = B_{(0)} \cup \Lambda \times \mathbb{N}_m \times B^n, \qquad \mathbf{S}_{(0)} = B^+, \quad \mathbf{S}_{(1)} = \langle B_{(1)}; \Sigma_{(1)} \rangle,$$

where:

B^n is the n-th cartesian power of B, $B^+ = \bigcup\{B^k | k \geq 1\}$ is a free semigroup with a basis B,

$\mathbf{S}_{(1)}$ is a semigroup given by the presentation $\langle B_{(1)}; \Sigma_{(1)} \rangle$ in the class of semigroups, where the set $\Sigma_{(1)}$ of defining relations is given by

$$\Sigma_{(1)} = \{(\alpha,1,x) \cdot \ldots \cdot (\alpha,m,x) = (\beta,1,x) \cdot \ldots \cdot (\beta,m,x) | \alpha,\beta \in \Lambda,\ x \in B^n\}.$$

Clearly, we can assume that

(1) $B_{(1)}$ is a generating subset of $\mathbf{S}_{(1)}$, and $B^+ = \mathbf{S}_{(0)}$ is a subsemigroup of $\mathbf{S}_{(1)}$.

Moreover, the following statement is evident, as well.

(2) If $y, z \in B^+_{(1)}$, then $y = z$ in $\mathbf{S}_{(1)}$ iff $y = z$ in $B^+_{(1)}$ or $y = w_0 \cdot y_1 \cdot w_1 \cdot \ldots \cdot y_t \cdot w_t$, $z = w_0 \cdot z_1 \cdot w_1 \cdot \ldots \cdot z_t \cdot w_t$, where $w_i \in B^*$, $y_j, z_j \in B^+_{(1)}$ and $y_j = z_j \in \Sigma_{(1)}$ for every $j \in \mathbb{N}_t$. (As usually, we denote by X^* a free monoid with a basis X, i. e. $X^* = X^+ \cup \{1\}$, where $1\ (\notin X^+)$ is the identity of X^*.)

Now, assume that p is a positive integer and that the following sequences are defined:

$B_{(0)} \subset B_{(1)} \subset \ldots \subset B_{(p)}$ — a sequence of sets,

$\mathbf{S}_{(0)} < \mathbf{S}_{(1)} < \ldots < \mathbf{S}_{(p)}$ — a sequence of semigroups,

$\Sigma_{(1)} \subset \Sigma_{(2)} \subset \ldots \subset \Sigma_{(p)}$ — a sequence of semigroup defining relations,

satisfying the following conditions:

$B_{(0)} = B$, $\mathbf{S}_{(0)} = B^+$, $B_{(i)}$ is a generating subset of $\mathbf{S}_{(i)}$, and

$$\mathbf{S}_{(i)} = \langle B_{(i)}; \Sigma_{(i)} \rangle, \tag{2.1}$$

$$B_{(j+1)} = B_{(j)} \cup \Lambda \times \mathbb{N}_m \times (B_{(j)})_n, \tag{2.2}$$

$$\Sigma_{(j+1)} = \{(\alpha,1,x) \cdot \ldots \cdot (\alpha,m,x) = (\beta,1,x) \cdot \ldots \cdot (\beta,m,x) | \alpha,\beta \in \Lambda,\ x \in (B_{(j)})_n\} \cup \Sigma_{(j)} \tag{2.3}$$

for every i, j: $0 \leq i \leq p$, $0 \leq j < p$.

Define $B_{(p+1)}$, $\mathbf{S}_{(p+1)}$, $\Sigma_{(p+1)}$ in such a way that (2.1), (2.2) and (2.3) hold for $i = p+1$, $j = p$.

Then we have:

(1') $B_{(p+1)}$ is a generating subset of $\mathbf{S}_{(p+1)}$ and $\mathbf{S}_{(p)}$ is a subsemigroup of $\mathbf{S}_{(p+1)}$.

It is also clear that replacing $B_{(1)}$, $\mathbf{S}_{(1)}$, $\Sigma_{(1)}$ in (2) by $B_{(i)}$, $\mathbf{S}_{(i)}$, $\Sigma_{(i)}$ respectively, then the corresponding generalization (2') of (2) also holds.

Define $[B;\Lambda]$, Σ and $\mathbf{S}$ as follows:

$$[B;\Lambda]=\bigcup\{B_{(p)}|\,p\geq 0\},\qquad \Sigma=\bigcup\{\Sigma_{(p)}|\,p\geq 0\},\qquad \mathbf{S}=\langle[B;\Lambda];\Sigma\rangle.$$

Then it can be easily shown that

(1") $[B;\Lambda]$ is a generating subset of $\mathbf{S}$, and $\mathbf{S}_{(p)}$ is a subsemigroup of $\mathbf{S}$, and moreover $\mathbf{S}=\bigcup\{\mathbf{S}_{(p)}|\,p\geq 0\}$.

A corresponding statement (2") obtained in such a way that $B_{(1)}$, $\mathbf{S}_{(1)}$, $\Sigma_{(1)}$ in (2) are replaced by $[B;\Lambda]$, $\mathbf{S},\Sigma$ respectively, is also true.

Finally, we define an object $\mathbf{F(B;\Lambda)}=(\mathbf{S};[B;\Lambda];f)\in Sem(n,m)$ as follows:

$$f(x)=(\alpha_0,1,x)(\alpha_0,2,x)\cdot\ldots\cdot(\alpha_0,m,x) \tag{2.4}$$

for every $x\in[B;\Lambda]_n$, where α_0 is a fixed element of Λ.

It follows from the definition of $\mathbf{S}$ that (2.4) holds for every $\alpha\in\Lambda$.

The next statement shows that $\mathbf{F(B;\Lambda)}$ has the desired properties.

Proposition 2.4. $\mathbf{F(B;\Lambda)}$ *is a weakly free object in* $Sem(n,m)$, *and* B *is the unique weak basis of* $\mathbf{F(B;\Lambda)}$.

Proof. Let P be a strong subgroupoid of $\mathbf{F(B;\Lambda)}$ such that $B\subseteq P$. Suppose that $B_{(t)}\subseteq P$ and take $u\in B_{(t+1)}\setminus B_{(t)}$. Then $u=(\alpha,i,x)$ for some $\alpha\in\Lambda$, $i\in\mathbb{N}_m$, $x\in(B_{(t)})_n$, i.e. $x=u_1\cdot u_2\cdot\ldots\cdot u_n$, where $u_\lambda\in B_{(t)}\subseteq P$. Then

$$f(x)=(\alpha,1,x)\cdot\ldots\cdot(\alpha,m,x)\in P_m,$$

which implies $u=(\alpha,i,x)\in P$. Thus $[B;\Lambda]\subseteq P$ $(\subseteq[B;\Lambda])$, i.e. B is a weakly generating subset of $\mathbf{F(B;\Lambda)}$.

It is clear that, if b is a fixed element of B, then $[B;\Lambda]\setminus\{b\}=P$ is a strong subgroupoid of $\mathbf{F(B;\Lambda)}$, and therefore we obtain that B is a subset of every weakly generating subset of $\mathbf{F(B;\Lambda)}$.

Thus, we have to show that B is free in $\mathbf{F(B;\Lambda)}$.

Let $\varphi: B\to Q'$ be a mapping, where $\mathbf{Q'}=(\mathbf{S'};Q';f')\in Sem(n,m)$. We will show that there is a homomorphism $\xi:\mathbf{F(B;\Lambda)}\to\mathbf{Q'}$ which extends φ.

Put $\xi_0=\varphi$ and suppose that for all $p\leq t$, we have well defined mappings $\xi_p:B_{(p)}\to Q'$ such that the following three conditions hold when $p<t$:

(a) ξ_p is a restriction of ξ_{p+1}.

(b) $f(u_1\cdot\ldots\cdot u_n)=(\alpha,1,x)\cdot\ldots\cdot(\alpha,m,x)\Rightarrow$

$\Rightarrow f'(\xi_p(u_1)\cdot\ldots\cdot\xi_p(u_n))=\xi_{p+1}(\alpha,1,x)\cdot\ldots\cdot\xi_{p+1}(\alpha,m,x)$ for every $u_\lambda\in B_{(p)}$.

(c) $\xi_{p+1}(\alpha,i,x)=\xi_{p+1}(\beta,i,x)$ for every $(\alpha,i,x),(\beta,i,x)\in B_{(p+1)}$.

(Note that (a), (b), (c) hold trivially when $t=0$.)

Now define $\xi_{t+1}:B_{(t+1)}\to Q'$ as follows. ξ_t is the restriction of ξ_{t+1} over $B_{(t)}$. Let $u\in B_{(t+1)}\setminus B_{(t)}$, i.e. $u=(\alpha,i,x)$ for some $\alpha\in\Lambda$, $i\in\mathbb{N}_m$, $x=u_1\cdot\ldots\cdot u_n$, where $u_\lambda\in B_{(t)}$. (Note that then we also have $(\beta,j,x)\in B_{(t+1)}\setminus B_{(t)}$, for every $\beta\in\Lambda$, $j\in\mathbb{N}_m$.) Put

$$a'_\lambda=\xi_t(u_\lambda)\in Q',\ \text{for } \lambda\in\mathbb{N}_m.$$

Then there are $b'_1,\ldots,b'_m\in Q'$, such that

$$f'(a'_1\cdot\ldots\cdot a'_n)=b'_1\cdot\ldots\cdot b'_m. \tag{2.5}$$

We fix a sequence $b'_1,\ldots,b'_m\in Q$ such that (2.5) holds and put

$$\xi_{t+1}(\beta,j,x)=b'_j\ \text{for all } \beta\in\Lambda,\ j\in\mathbb{N}_m.$$

ξ_{t+1} is well defined, since if $x=u_1\cdot\ldots\cdot u_n=v_1\cdot\ldots\cdot v_n$ is an equality in $\mathbf{S}$, then by (2") and (c) we have $\xi_t(u_\lambda)=\xi_t(v_\lambda)$ for every $\lambda\in\mathbb{N}_n$. It is clear that ξ_{t+1} satisfies the properties (a), (b), (c), which implies that if we put $\xi=\bigcup_{p\geq 0}\xi_p$, then ξ is a homomorphism and ξ is an extension of φ. □

It follows from Proposition 2.3 that there exists a free subgroupoid of $\mathbf{F(B;\Lambda)}$ with a basis B. We will show that for every $\alpha\in\Lambda$ there exists a free subgroupoid $[B;\alpha]$ of $\mathbf{F(B;\Lambda)}$ with a basis B.

Namely, if α is a given element of Λ, then we define a subset $[B;\alpha]$ of $[B;\Lambda]$ as follows:

$$C_{(0)}=B,\ \ D_{(i)}=\bigcup\{(\alpha,j,x)|\,j\in\mathbb{N}_m,\ x\in(C_{(i)})_n\},\ \ C_{(i+1)}=C_{(i)}\cup D_{(i)},\ \ [B;\alpha]=\bigcup\{C_{(i)}|\,i\geq 0\}.$$

Proposition 2.5. *For every $\alpha\in\Lambda$, $[B;\alpha]$ is a free subobject of* $\mathbf{F(B;\Lambda)}$ *with a basis B. If $\alpha\neq\beta$, then $[B;\alpha]\cap[B;\beta]=B$.*

Proof. $[B;\alpha]$ is a subobject of $\mathbf{F(B;\Lambda)}$ since $[B;\alpha]\subseteq[B;\Lambda]$ and $u_\lambda\in[B;\alpha]\Rightarrow f(u_1\cdot\ldots\cdot u_n)=(\alpha,1,x)\cdot\ldots\cdot(\alpha,m,x)\in[B;\alpha]_m$, where $x=u_1\cdot\ldots\cdot u_n$. It is clear that $B\subseteq[B;\alpha]$, and suppose that P is a subobject of $[B;\alpha]$ such that $B\subseteq P$. Let $C_{(t)}\subseteq P$ and $u\in C_{(t+1)}\setminus C_{(t)}$. Then $u=(\alpha,i,x)$ for some $i\in\mathbb{N}_m$, $x\in(C_{(t)})_n$, and

$$f(x)=(\beta,1,x)\cdot\ldots\cdot(\beta,m,x)\in P_m.$$

Then there are $u_1,\ldots,u_m\in P$ such that $(\beta,1,x)\cdot\ldots\cdot(\beta,m,x)=u_1\cdot\ldots\cdot u_m$ holds in $\mathbf{S}$, and the definition of Σ implies that this is possible only if $u_j=(\alpha,j,x)$, i.e. $u\in P$.

Thus B is a generating subset of $[B;\alpha]$. To show that B is free in $[B;\alpha]$ one can use the same construction as in Proposition 2.4, with needed restriction, certainly. □

Remark. Note that the assumption $m\geq 2$ is essential only in the last conclusion $\alpha\neq\beta\Rightarrow[B;\alpha]\cap[B;\beta]=B$. Namely, in the case $m=1$, we have $[B;\alpha]=[B;\beta]=[B;\Lambda]$, for every $\alpha,\beta\in\Lambda$.

As a summary of the above results we got the following

Theorem 2.6. *Let n and m be two positive integers, such that $m\geq 2$, and let B be a nonempty set.*

(i) *For each cardinality $\sigma\geq\max\{|B|,\aleph_0\}$ there is a weakly free object $\mathbf{Q}=(\mathbf{S};Q;f)$ with a weak basis B, such that $|Q|=\sigma$. Moreover, σ is the cardinality of the collection of free subobjects of* $\mathbf{Q}$ *with the same basis B.*

(ii) *There exist nonisomorphic weakly free objects with same weak basis B.* □

In the next example we will show that there exist weakly free objects which are not isomorphic to any object of the form $\mathbf{F(B;\Lambda)}$.

Example. Let $A=\{a_1,\ldots,a_m\}$ be a given set such that $|A|=m\geq 2$, $A\cap[B;\Lambda]=\emptyset$. Fix elements $x_0\in[B;\Lambda]_n$, $\alpha_0\in\Lambda$ and define a semigroup $\mathbf{T}$ generated by $[B;\Lambda]\cup A$ and an object $\mathbf{G}=(\mathbf{T};[B;\Lambda]\cup A;g)$ as follows:

$$\mathbf{T}=\langle[B;\Lambda]\cup A;\{a_1\cdot\ldots\cdot a_m=(\alpha_0,1,x_0)\cdot\ldots\cdot(\alpha_0,m,x_0)\}\cup\Sigma\rangle,$$

$$g(u_1\cdot\ldots\cdot u_n)=f(u_1'\cdot\ldots\cdot u_n'),$$

where

$$u'=\begin{cases}u, & \text{if } u\in[B;\Lambda],\\ (\alpha_0,i,x_0), & \text{if } u=a_i\in A.\end{cases}$$

Then B is a weakly generating subset of $\mathbf{G}$, since

$$g(x_0)=f(x_0)=(\alpha_0,1,x_0)\cdot\ldots\cdot(\alpha_0,m,x_0)=a_1\cdot\ldots\cdot a_m.$$

We can use the construction in Proposition 2.4 to show that B is free in $\mathbf{G}$. Namely, we can use ξ to get a mapping η by putting $\eta(a_i)=\xi(\alpha_0,i,x_0)$ and $\eta(u)=\xi(u)$ for $u\in[B;\Lambda]$. Thus, η will be a homomorphism from $\mathbf{G}$ into $\mathbf{Q}'$, which extends $\varphi: B\to Q'$.

In such a way we obtained a weakly free object $\mathbf{G}$, not isomorphic to $\mathbf{F(B;\Lambda)}$. As before, the weak basis B of $\mathbf{G}$ is uniquely determined. □

We note that the obtained results about weakly free objects in $Sem(n,m)$ are not complete. For example, in all above examples of weakly free objects, the weak bases are unique, but we do not know whether this is true for every weakly free object.

References

[1] P. M. Cohn: *Universal Algebra*. Harper & Row, New York, 1965.

[2] Ǵ. Čupona, N. Celakoski, S. Markovski and D. Dimovski: *Vector valued groupoids, semigroups and groups. "Vector Valued Semigroups and Groups"*. Mac. Acad. Sci. and Arts (1988) 1–78.

[3] Ǵ. Čupona and S. Markovski: *Polyadic operations induced by semigroups*. Matem. Bilten **13** (XXXIX), Skopje (1989) 15–24.

[4] Ǵ. Čupona and S. Markovski: *Vector valued groupoids induced by varieties of semigroups*. Matem. Bilten **14** (XL), Skopje (1990) 27–38.

[5] Ǵ. Čupona, A. Samardžiski and N. Celakoski: *Fully commutative vector valued groupoids*. Proceedings of the Conference "Algebra and Logic" Sarajevo, 1987, 29–41.

[6] Ǵ. Čupona, D. Dimovski and A. Samardžiski: *Fully commutative vector valued groups*. Mac. Acad. Sci. and Arts, Contributions, Sect. of Math. and Techn. Sc. VIII/2, Skopje, 1987 (1989) 5–27.