

FREE OBJECTS IN PRIMITIVE VARIETIES OF n -GROUPOIDS

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G. Čupona and S. Markovski

Dedicated to the memory of Professor D. Kurepa

Abstract. A variety of n -groupoids (i.e. algebras with one n -ary operation f) is said to be a primitive n -variety if it is defined by a system of identities of the following form:

$$f(x_{i_1}, x_{i_2}, \dots, x_{i_n}) = f(x_{j_1}, x_{j_2}, \dots, x_{j_n}) \quad (*)$$

Here we give a convenient description of free objects in primitive n -varieties, and several properties of free objects are also established.

1. Introduction. Identities of the form $(*)$ are called primitive n -identities, where we take n to be a fixed positive integer, and i_λ, j_μ are positive integers. We do not make any distinction between two equivalent identities, and that is the reason why we assume $1 \leq i_\nu, j_\nu \leq 2n$. A set Σ of primitive n -identities is said to be complete if it contains every primitive n -identity which is a consequence of Σ . Everywhere in this paper we suppose that Σ is a complete system of primitive n -identities, and we also take $n \geq 2$, since for $n = 1$ the only nontrivial primitive 1-identity is $f(x) = f(y)$, which gives rise to constant unars.

The main results obtained here are the construction of free Σ -objects with given basis B and the following theorems, which are corollaries of the obtained construction.

THEOREM A. *A free Σ -object has a unique basis.* \square

THEOREM B. *Every subobject of a free Σ -object is a free Σ -object as well.* \square

For any identity $(*)$ we put $I = \{i_1, \dots, i_n\}$, $J = \{j_1, \dots, j_n\}$.

THEOREM C. *Assume that there is an identity $(*)$ in Σ such that $I \cap J = \emptyset$. If $k \in \{1, 2, \dots, n-1\}$ is the largest integer such that $(*)$ is in Σ for $I = \{1\}$ and for every J with $|J| \leq k$, then any free Σ -object with rank k has a subobject with infinite rank.* \square

THEOREM D. *For every identity $(*)$ in Σ let $I \cap J \neq \emptyset$ and assume that if $(*)$ is in Σ for $I = \{1, 2, \dots, n\}$, then $|J| \geq 2$. Then every free Σ -object has a subobject with infinite rank.* \square

THEOREM E. *The word problem is solvable in any primitive n -variety.* \square

2. Complete sets of primitive n -identities. As we already mentioned in section 1, we assume that in $(*)$ we have $1 \leq i_\nu, j_\nu \leq 2n = m$ for each ν . In such a way the primitive n -identities can be considered as transformations of the set $M = \{1, 2, \dots, m\}$, i.e. as elements of the set $\mathcal{T} = M^m (= \{\varphi | \varphi: M \rightarrow M\})$. Next, in this paper we will not make any distinction between the sets M^m and $M^n \times M^n$, where $M^n = \{\psi | \psi: \{1, 2, \dots, n\} \rightarrow M\}$. Namely, if $\varphi \in M^m$ and $\varphi_L, \varphi_R \in M^n$ are defined by

$$\varphi_L(i) = \varphi(i), \quad \varphi_R(i) = \varphi(n+i)$$

for each $i \in \{1, 2, \dots, n\}$, then (φ_L, φ_R) will be considered as another notation of φ .

We stress again that we suppose here and further on that Σ denotes a complete set of primitive n -identities, where $n \geq 2$ is a given integer. By the above agreement, we also have that $\Sigma \subseteq \mathcal{T}$.

Every subset Λ of \mathcal{T} induces a relation \sim_Λ on M^n defined by

$$\varphi \sim_\Lambda \psi \Leftrightarrow (\varphi, \psi) \in \Lambda.$$

The following completeness theorem is a consequence of a result from [2]:

PROPOSITION 2.1. *A subset Λ of \mathcal{T} is complete iff it satisfies the following conditions:*

- (i) \sim_Λ is an equivalence relation on M^n ;
- (ii) Λ is a left ideal in \mathcal{T} , i.e. $\mathcal{T} \circ \Lambda \subseteq \Lambda$, where \circ denotes the usual superposition of transformations. \square

The following property (shown in [2]) will be used in the next section:

PROPOSITION 2.2. *Let $\xi, \eta \in \Sigma$ be such that $\ker \xi_R = \ker \eta_L$, and denote by $T(\xi, \eta)$ the set of all elements $\zeta \in \mathcal{T}$ which satisfy the following conditions: $\zeta_L = \xi_L$ and*

$$\xi(i) = \xi(k+n), \quad \eta(k) = \eta(j+n) \Rightarrow \zeta(i) = \zeta(j+n)$$

for every $i, k, j \in \{1, 2, \dots, n\}$. Then $T(\xi, \eta) \neq \emptyset$ and $T(\xi, \eta) \subseteq \Sigma$ (and, furthermore, $\mathcal{T} \circ T(\xi, \eta) \subseteq \Sigma$). \square

Given any complete set Σ of primitive n -identities, by $\Sigma[M]$ we denote the

quotient set M^n / \sim_Σ , and if $\varphi \in M^n$, then by $[\varphi] \in \Sigma[M]$ we denote the corresponding class of equivalent elements. (Further on, we will write simply \sim instead of \sim_Σ .)

For any $\mathbf{i} \in M$, let $i \in M^n$ be defined by $\mathbf{i}(\nu) = i$ for each $\nu \in \{1, 2, \dots, n\}$. We say that Σ is with constant if $[1] = [2]$.

If $\varphi \in M^n$, then the set $\{\varphi(1), \dots, \varphi(n)\}$ is called the content of φ , and will be denoted by $\text{cnt}(\varphi)$.

PROPOSITION 2.3. *The following conditions are equivalent:*

- (i) Σ is with constant;
- (ii) $[\mathbf{i}] = [\mathbf{j}]$ for any $\mathbf{i}, \mathbf{j} \in M$;
- (iii) there exist $\varphi, \eta \in M^n$ such that $[\varphi] = [\eta]$ and the contents of φ and η are disjoint. \square

If Σ is with constant, then any element of $[\mathbf{i}]$ is called a Σ -constant; Σ is said to be with absolute constant if $\Sigma[M]$ is a singleton. Denote by ε the element of M^n defined by $\varepsilon(\nu) = \nu$ for each $\nu \in \{1, 2, \dots, n\}$.

PROPOSITION 2.4. *The following conditions are equivalent:*

- (i) Σ is with absolute constant;
- (ii) $\varphi \sim \eta$ for any $\varphi, \eta \in M^n$;
- (iii) there is a $\varphi \in M^n$ such that $\varepsilon \sim \varphi$ and ε and φ have disjoint contents. \square

PROPOSITION 2.5. *If $\varphi \in M^n$ is not a Σ -constant, then there is an $\eta \in [\varphi]$ such that $\text{cnt}(\eta)$ is a subset of $\text{cnt}(\psi)$ for any $\psi \in [\varphi]$.*

(Then we say that η is a minimal member of $[\varphi]$.)

Proof. Since $A = \{\text{cnt}(\xi) \mid \xi \in [\varphi]\}$ is a finite set, there is an $\eta \in [\varphi]$ such that $\text{cnt}(\eta)$ is a minimal member in A . Assume that $\text{cnt}(\eta)$ and $\text{cnt}(\eta')$ are different minimal members in A . Then $\text{cnt}(\eta) \cap \text{cnt}(\eta') \neq \emptyset$, since φ is not a Σ -constant. Let $i \in \text{cnt}(\eta) \cap \text{cnt}(\eta')$ and let $j \in \text{cnt}(\eta') \setminus \text{cnt}(\eta)$. Define $\zeta \in \mathcal{T}$ by $\zeta(j) = i$ and $\zeta(k) = k$ for any $k \neq j$. Then $\zeta \circ (\eta, \eta') = (\eta, \eta'') \in \Sigma$ for some $\eta'' \in M^n$ such that $\text{cnt}(\eta'') = \text{cnt}(\eta') \setminus \{j\}$. \square

Now we define the notion of the Σ -content of an element $\varphi \in M^n$, denoted by $\text{cnt}_\Sigma(\varphi)$, as follows. We put $\text{cnt}_\Sigma(\varphi) = \emptyset$ if φ is a Σ -constant, and $\text{cnt}_\Sigma(\varphi) = \text{cnt}(\eta)$ if φ is not a Σ -constant and η is a minimal member of $[\varphi]$. Note that $\xi \sim \varphi$ implies $\text{cnt}_\Sigma(\xi) = \text{cnt}_\Sigma(\varphi)$.

PROPOSITION 2.6. *There exists a $\varphi \in M^n$ such that $\text{cnt}_\Sigma(\varphi)$ is a singleton iff Σ is without constant. \square*

Σ is said to be essentially k -ary iff $|\text{cnt}_\Sigma(\varepsilon)| = k$.

If Σ is with constant, then the order of the constant of Σ is said to be k iff $\text{cnt}_\Sigma(\varphi) = \emptyset$ for each $\varphi \in M^n$ such that $|\text{cnt}(\varphi)| \leq k$, and k is the largest such integer. Therefore we have:

PROPOSITION 2.7. *The following statements are equivalent:*

- (i) Σ is with absolute constant;
- (ii) Σ is with constant of order n . \square

3. Σ -objects. Let A be a nonempty set and let Σ be a complete set of primitive n -identities. Define a relation $\sim_{\Sigma, A}$ (shortly denoted by \sim_A) on the set $A^n (= \{\mathbf{a} \mid \mathbf{a}: \{1, \dots, n\} \rightarrow A\})$ as follows:

$$\mathbf{a} \sim \mathbf{b} \iff (\exists \xi \in \Sigma) \ker \xi = \ker(\mathbf{a}, \mathbf{b})$$

where $\mathbf{a}, \mathbf{b} \in A^n$ and $(\mathbf{a}, \mathbf{b}) \in A^m$ is defined as in the preceding section, i.e. $(\mathbf{a}, \mathbf{b})(i) = \mathbf{a}(i)$, $(\mathbf{a}, \mathbf{b})(i+n) = \mathbf{b}(i)$, for each $i \in \{1, 2, \dots, n\}$.

The following statement is a corollary from Proposition 1.1 (and its generalization as well):

PROPOSITION 3.1. (i) \sim_A is an equivalence relation. (ii) If $\mathbf{a} \sim_A \mathbf{b}$, \mathbf{c} is a transformation of A and $\mathbf{c} \circ (\mathbf{a}, \mathbf{b}) = (\mathbf{a}', \mathbf{b}')$, then $\mathbf{a}' \sim_A \mathbf{b}'$. \square

Proof. We will give only a sketch of the proof, and we will use the fact that Σ is a complete set of identities. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in A^n$.

- (i) Then for suitably chosen $\varphi \in \mathcal{T}$ we have $\ker \varphi(\varepsilon, \varepsilon) = \ker(\mathbf{a}, \mathbf{a})$, and also if $\ker \xi = \ker(\mathbf{a}, \mathbf{b})$, then $\ker(\xi_R, \xi_L) = \ker(\mathbf{b}, \mathbf{a})$. The transitivity follows by using Proposition 2.2.

- (ii) If $\ker \xi = \ker(\mathbf{a}, \mathbf{b})$ and $\mathbf{c} \circ (\mathbf{a}, \mathbf{b}) = (\mathbf{a}', \mathbf{b}')$, then there is a $\varphi \in \mathcal{T}$ such that $\ker \varphi \xi = \ker(\mathbf{a}', \mathbf{b}')$, and $\xi \in \Sigma$ implies $\varphi \xi \in \Sigma$ by Proposition 2.1. \square

We denote by $\Sigma[A]$ the quotient set A^n / \sim_A and by $[\mathbf{a}]$ the class of equivalent

elements of $\mathbf{a} \in A^n$. (So, $[\mathbf{a}] = [\mathbf{b}]$ iff $\mathbf{a} \sim_A \mathbf{b}$.) If $A = M = \{1, 2, \dots, m\}$, then \sim_A and \sim have the same meaning as in section 2.

Proposition 2.2–2.6 have obvious generalizations, and we make a summary below.

(1) $|\Sigma[A]| = 1$ iff one of the following cases appears: 1.1) $|A| = 1$; 1.2) Σ is with absolute constant; 1.3) $|A| \leq k$ and Σ is with constant of order k .

(2) If $\mathbf{a} \in A^n$, then the set $\text{cnt}(\mathbf{a}) = \{\mathbf{a}(1), \dots, \mathbf{a}(n)\}$ is called the content of \mathbf{a} . If Σ is with constant and $|\text{cnt}(\mathbf{a})| = 1$, then the class of equivalent elements $[\mathbf{a}]$ will be denoted by $\mathfrak{o}(\notin A)$ and called the zero of $\Sigma[A]$. Then we also say that the Σ -content of \mathfrak{o} is empty, and we denote it by $\text{cnt}_\Sigma(\mathfrak{o}) = \emptyset$; moreover, for each $\mathbf{c} \in \mathfrak{o}$ we put $\text{cnt}_\Sigma(\mathbf{c}) = \emptyset$. Let $\mathbf{b} \in A^n$. If either Σ is without constant or $[\mathbf{b}] \neq \mathfrak{o}$, then in the family of sets $\{\text{cnt}(\mathbf{c}) | \mathbf{c} \in [\mathbf{b}]\}$ there is the least member which will be denoted by $\text{cnt}_\Sigma[\mathbf{b}]$ and called the Σ -content of $[\mathbf{b}]$; in this case we also let $\text{cnt}_\Sigma(\mathbf{c}) = \text{cnt}_\Sigma[\mathbf{b}]$ for each $\mathbf{c} \in [\mathbf{b}]$. And, if $\mathbf{d} \in [\mathbf{b}]$ is such that $\text{cnt}(\mathbf{d}) = \text{cnt}_\Sigma[\mathbf{b}]$, then we say that \mathbf{d} is a minimal member of $[\mathbf{b}]$. (We note that $[\mathbf{b}]$ can contain distinct minimal members.)

(3) If Σ is with constant then $|\text{cnt}_\Sigma[\mathbf{a}]| \geq 2$ for each $[\mathbf{a}] \neq \mathfrak{o}$, but if Σ is without constant then $|\text{cnt}_\Sigma[\mathbf{a}]| = 1$ for every $\mathbf{a} \in A^n$ such that $|\text{cnt}(\mathbf{a})| = 1$. If Σ is essentially unary then $|\text{cnt}_\Sigma[\mathbf{a}]| = 1$ for every $\mathbf{a} \in A^n$.

(4) If $A \subseteq B$ then the canonical mapping from $\Sigma[A]$ into $\Sigma[B]$ is injective, and then we can assume that $\Sigma[A] \subseteq \Sigma[B]$, in the following sense: if $[\mathbf{a}] \in \Sigma[B]$ and $\text{cnt}_\Sigma[\mathbf{a}] \subseteq A$, then we take $[\mathbf{a}] \in \Sigma[A]$ as well.

An algebra (A, f) with n -ary operation f (i.e. an n -groupoid) is called a Σ -object if it satisfies all the identities belonging to Σ .

PROPOSITION 3.2. *An n -groupoid (A, f) is a Σ -object iff*

$$\mathbf{a} \sim_A \mathbf{b} \Rightarrow f(\mathbf{a}) = f(\mathbf{b})$$

for every $\mathbf{a}, \mathbf{b} \in A^n$. \square

Denote by $\text{nat}(\sim_A)$ the natural mapping $\mathbf{a} \mapsto [\mathbf{a}]$ from A^n into $\Sigma[A]$. Then by Proposition 3.2 we have:

PROPOSITION 3.3. *An n -groupoid (A, f) is a Σ -object iff there is a unique mapping $\underline{f}: \Sigma[A] \rightarrow A$ such that $\underline{f} \circ \text{nat}(\sim_A) = f$. (Certainly, the existence of such a mapping \underline{f} implies its uniqueness.) \square*

Now we have a more convenient alternative definition of a Σ -object. Namely, if \underline{f} is a mapping from $\Sigma[A]$ into A , then the pair (A, \underline{f}) is called a Σ -object with carrier A and operation \underline{f} . Further on, by a Σ -object we will understand the kind of structure we have just defined. Thus, for subobjects and homomorphisms we have the following characterizations:

PROPOSITION 3.4. *If $\mathbf{A} = (A, \underline{f})$ is a Σ -object and $C \subseteq A$, then C is a subobject of \mathbf{A} iff $\underline{f}(\Sigma[C]) \subseteq C$. \square*

Thus, any subobject of a Σ -object is a Σ -object too.

PROPOSITION 3.5. *Let $\mathbf{A} = (A, \underline{f})$ and $\mathbf{B} = (B, \underline{g})$ be Σ -objects, and let $h: A \rightarrow B$ be a mapping. Then h induces a unique mapping $\underline{h}: \Sigma[A] \rightarrow \Sigma[B]$ such that $\underline{h} \circ \text{nat}(\sim_A) = \text{nat}(\sim_B) \circ h$, and h is a homomorphism from \mathbf{A} into \mathbf{B} iff $h \circ \underline{f} = \underline{g} \circ \underline{h}$. \square*

(We note that $h: A \rightarrow B$ induces a mapping $h^{(n)}: A^n \rightarrow B^n$ such that $[\mathbf{a}] = [\mathbf{b}]$ in $\Sigma[A]$ implies $[h^{(n)}(\mathbf{a})] = [h^{(n)}(\mathbf{b})]$ in $\Sigma[B]$, and then $\underline{h}([\mathbf{a}]) = [\underline{h}^{(n)}(\mathbf{a})]$ for each $\mathbf{a} \in A^n$.)

The notion of a partial Σ -object can be defined as follows. Let A be a nonempty set, \mathcal{D} a subset of $\Sigma[A]$ and \underline{f} a mapping from \mathcal{D} into A . Then we say that the triple $(A, \mathcal{D}, \underline{f})$ is a partial Σ -object. It can be easily seen that this definition is compatible with Evans' definition of partial algebras in a variety of algebras (see [3], where the words "incomplete" and "a class of algebras \mathcal{V} " are used instead of "partial" and "a variety \mathcal{V} "). Furthermore, if $(A, \mathcal{D}, \underline{f})$ is a given partial Σ -object and q a fixed element of A , then if we define $\underline{g}: \Sigma[A] \rightarrow A$ by

$$\underline{g}([\mathbf{a}]) = \begin{cases} \underline{f}([\mathbf{a}]), & \text{if } [\mathbf{a}] \in \mathcal{D} \\ q, & \text{if } [\mathbf{a}] \in \Sigma[A] \setminus \mathcal{D}' \end{cases}$$

then (A, \underline{g}) is a Σ -object which is an extension of $(A, \mathcal{D}, \underline{f})$. Now we can apply the well known Evans' result [3, p. 68] "if \mathcal{V} is a class of algebras having the property

that any incomplete \mathcal{V} -algebra can be embedded in a \mathcal{V} -algebra, then the word problem can be solved for this class" to obtain the proof of Theorem E of section 1.

4. A construction of free Σ -objects. Here we will give a construction of free Σ -objects with basis B , where B is a given nonempty set. Let $(B_p | p \geq 0)$ be a sequence of sets defined inductively as follows:

$$B_0 = B, \quad B_{p+1} = B_p \cup \Sigma[B_p],$$

and let

$$F(\Sigma, B) = \bigcup (B_p | p \geq 0).$$

(We will write simply F instead of $F(\Sigma, B)$, when Σ and B are known.) By induction on p one can easily prove that $\Sigma[F] = F \setminus B$.

If $u \in F$ and if p is the least number such that $u \in B_p$, then we say that p is the hierarchy of u and write $\chi(u) = p$. It is clear that if Σ is with constant, then $\chi(\mathbf{o}) = 1$.

PROPOSITION 4.1. *Let $u \in F$ and let u not be a constant. Then $\chi(u) = p + 1$ iff $\text{cnt}_\Sigma(u) = \{v_1, v_2, \dots, v_k\}$ is such that $\chi(v_i) \leq p$ for each i and $\chi(v_j) = p$ for some j ($i, j \in \{1, 2, \dots, k\}$).* \square

Define an operation $\underline{f}: \Sigma[F] \rightarrow F$ by $\underline{f}(u) = u$ for each $u \in \Sigma[F]$. Then we have:

PROPOSITION 4.2. *(F, \underline{f}) is a Σ -object generated by the set B .* \square

Let (C, \underline{g}) be an arbitrary Σ -object and let $h: B \rightarrow C$ be a mapping. Put $h_0 = h$ and suppose that $h_r: B_r \rightarrow C$ is a well defined mapping for each $r \leq p$ in such a way that h_r is an extension of h_{r-1} , and if $r > 0$, $\chi(u) = r$, then $h_r(u) = \underline{g} \circ \underline{h}_{r-1}(u)$, where $\underline{h}_{r-1}: \Sigma[B_{r-1}] \rightarrow \Sigma[C]$ is defined as in Proposition 3.5. Now define $h_{p+1}: B_{p+1} \rightarrow C$ to be the extension of h_p such that $h_{p+1}(u) = \underline{g} \circ \underline{h}_p(u)$ for each u with $\chi(u) = p + 1$. (Note that if $\chi(u) = p + 1$, then $u \in \Sigma[B_p]$, and thus $\underline{h}_p(u) \in \Sigma[C]$ is well defined by Proposition 3.5.) In such a way we have defined a chain of mappings $(h_p | p \geq 0)$, and its union $\bar{h} = \bigcup (h_p | p \geq 0)$ is an extension of h and a homomorphism from (F, \underline{f}) into (C, \underline{g}) as well. Thus we have the following

THEOREM 4.3. *If B is a nonempty set, then (F, \underline{f}) is a free object with basis B .* \square

The preceding construction of free Σ -objects is somewhat obscure, but in some cases it can be considerably simplified.

Example 4.4. If Σ is with constant and $a, b \in B$, then we have $[a^n] = [b^n] = \mathbf{o}$, where \mathbf{o} is the zero of F . (Here, and later on, $a^n: i \mapsto a$ for each $a \in A$, $i \in \{1, \dots, n\}$.) Clearly, $\mathbf{o} \in B_1 \setminus B$ and if Σ is with absolute constant, then $F = B \cup \{\mathbf{o}\}$ and $\underline{f}(u) = \mathbf{o}$ for each $u \in \Sigma[B \cup \{\mathbf{o}\}]$. Therefore, if Σ is with absolute constant, then every constant n -groupoid is freely generated by the set of elements distinct from the constant (i.e. \mathbf{o}). We have the same result if Σ is with constant, of order k and $|B| < k$. (Moreover, if Σ is with constant, then any one-element groupoid can be considered as free Σ -object with empty basis.) \square

Example 4.5. Assume that Σ is essentially unar, i.e. for each $\varphi \in M^n$ there is an $i \in \{1, 2, \dots, n\}$ such that $(\varphi, j) \in \Sigma$ for $j = \varphi(i)$. Then the class of Σ -objects can be viewed as the class of unars. Namely, if (G, h) is a unar and if we define a mapping $\underline{g}: \Sigma[G] \rightarrow G$ by $\underline{g}(a) = h(a(i))$, then we get a Σ -object (G, \underline{g}) , and any Σ -object can be obtained in such a manner. Moreover, (G, \underline{g}) is a free Σ -object with basis B iff (G, h) is a free unar with basis B . \square

We note that a subunar of a finitely generated free unar is finitely generated too, and thus Example 4.5 shows that the assumptions of Theorem D are essential.

Example 4.6. Let $n = 3$ and let \mathcal{V} be a variety defined by the identities

$$f(x, x, x) = f(x, x, y) = f(y, y, y), \quad f(x, y, z) = f(y, x, z) = f(x, z, y).$$

If $B = \{b\}$, $\mathbf{o} \neq b$ and if we put $G = \{\mathbf{o}, b\}$ and $g(u, v, w) = \mathbf{o}$ for each $u, v, w \in G$, then (G, g) is a free object in \mathcal{V} with basis B of rank 1. Now, take $B = \{b, c\}$, $b \neq c$ and $\mathbf{o} \notin B$, and define the sets B_p inductively by

$$B_0 = B \cup \{\mathbf{o}\}, \quad B_{p+1} = B_p \cup \{\{u, v, w\} | u \neq v \neq w \neq u, u, v, w \in B_p\}$$

Let $H = \bigcup (B_p | p \geq 0)$ and let

$$h(u, v, w) = \begin{cases} \{u, v, w\}, & \text{if } u \neq v \neq w \neq u. \\ \mathbf{o}, & \text{otherwise} \end{cases}$$

Then $\mathbf{H} = (H, h)$ is a free object in \mathcal{V} with basis B . The subset D of H , where $D = \{d_i | i \geq 0\}$ and the elements d_i are defined inductively by $d_0 = \{\mathbf{o}, b, c\}$, $d_{p+1} =$

$\{o, b, d_p\}$ is a basis of infinite rank of the subobject \mathbf{L} of \mathbf{H} generated by D . \square

Example 4.7. There exist exactly 6 nonequivalent primitive 2-identities: $xy = xy$, $xy = yx$, $xy = xx$, $xy = yy$, $xx = yy$, $xy = zw$. (Here a usual notation of identities is used.) One can form 7 primitive 2-varieties, 6 of them being defined by a single identity of the above ones, and $\mathcal{V} = \text{Var}(\{xy = yx, xx = yy\})$. In the variety \mathcal{V} we can describe a free object with nonempty basis B by $F = \bigcup (B_p | p \geq 0)$, where $B_0 = B$, $B_1 = B \cup \{o\} \cup \{\{u, v\} | u, v \in B, u \neq v\}$, $o \notin B$, and $B_{p+1} = B_p \cup \{\{u, v\} | u, v \in B_p, u \neq v\}$ when $p \geq 1$. \square

5. Some properties of free Σ -objects. Here we will give proofs of Theorems A, B, C and D of section 1. Although one can prove these theorems by using an induction on hierarchy, we will rather use the ideas involved in [1].

Assume that $\mathbf{G} = (G, g)$ is a Σ -object. An element $a \in G$ is said to be prime in \mathbf{G} if $a \neq g([b])$ for any $[b] \in \Sigma[G]$. If Σ is with constant, then each element of \mathbf{G} is said to be an improper divisor of the zero $o \in \Sigma[G]$. If $c \in G$ is nonzero and nonprime element, then there is a $[b] \in \Sigma[G]$ such that $c = g([b])$, and let a be a minimal member of $[b]$. Then each element $d \in \text{cnt}(a) = \text{cnt}_{\Sigma}[a]$ is called a proper divisor of c . A sequence (finite or infinite) of elements a_1, a_2, \dots of G is said to be a divisor chain in \mathbf{G} iff for every $i > 1$ a_i is a proper divisor of a_{i-1} .

Now we have another characterization of free Σ -objects:

THEOREM 5.1. *A Σ -object $\mathbf{H} = (H, \underline{h})$ is a free Σ -object with a nonempty basis $B \subseteq H$ iff the following conditions hold:*

- (i) *B is the set of prime elements in \mathbf{H} .*
- (ii) *If $c \in H$ is nonprime, then there is a unique $[b] \in \Sigma[B]$ such that $c = \underline{h}([b])$.*
- (iii) *Every divisor chain in \mathbf{H} is finite.*

Proof. It is clear that (F, f) satisfies (i), (ii) and (iii).

Conversely, if \mathbf{H} satisfies (i), (ii) and (iii), then it is easy to show by induction on hierarchy that there is an isomorphism $g: (F, f) \rightarrow (H, \underline{h})$ such that $g(b) = b$ for each $b \in B$. \square

Now, Theorem A is a direct consequence of Theorem 5.1, for the set of prime elements of a free Σ -object is its unique basis. (We should emphasize here that we do not need Theorem 5.1 to prove Theorem A, since it follows directly from the definition of primitive n -identities.)

Assume that \mathbf{G} is a subobject of (F, f) . The set of prime elements in \mathbf{G} (considered as a Σ -object) is empty only if Σ is with zero and $G = \{o\}$, and then \mathbf{G} is free with an empty basis. If the set C of prime elements in \mathbf{G} is nonempty, then C is a basis of \mathbf{G} , since conditions (ii) and (iii) of Theorem 5.1 are hereditary. This completes the proof of Theorem B.

Now, let Σ be with constant of order $k < n$, and let $B = \{a_1, a_2, \dots, a_k\}$. Then $B_1 = B \cup \{o\}$ and $\text{cnt}_{\Sigma}(a_1 a_2 \dots a_k o^{n-k}) = \{a_1, a_2, \dots, a_k, o\}$. Consider the subset $C = \{c_1, c_2, \dots, c_p, \dots\}$ of F , where $c_1 = [a_1 \dots a_k o^{n-k}]$, $c_{p+1} = [a_1 \dots a_k c_p^{n-k}]$. Let Q be the subobject of (F, f) generated by C . Clearly, C is the set of prime elements in Q . (Namely, c_p is a divisor of c_{p+1} in F , but this does not hold in Q .) This completes the proof of Theorem C, since the conditions for Σ stated in Theorem C show that Σ is with constant of order k .

It remains to show Theorem D. First we note that the assumption in this Theorem can be expressed by $|\text{cnt}_{\Sigma}(\varepsilon)| = k \geq 2$. Take φ to be a minimal member in $[\varepsilon]$, and $i \in \text{cnt}_{\Sigma}(\varphi)$. Let B be a nonempty set, $b \in B$ and define a sequence a_1, a_2, \dots, a_n by $a_1 = b$, $a_{i+1} = [a_i^n]$ for $0 < i < n$, and an infinite sequence $c_1, c_2, \dots, c_p, \dots$ by $c_1 = a_n$, $c_{p+1} = [a_1 a_2 \dots a_{i-1} c_p a_{i+1} \dots a_n]$. Then $a_i \neq a_j$ for $i \neq j$ and $c_r \neq c_s$ for $r \neq s$. This implies that $C = \{c_r | r \geq 1\}$ is an infinite basis of the subobject Q of (F, f) generated by C .

REFERENCES

1. R.H. Bruck, *A Survey of Binary Systems*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958
2. G. Čupona, S. Markovski S., Ž. Popeska, *Primitive n -identities*, Contributions to General Algebra 9 (1995), 107-116
3. T. Evans, *The word problem for abstract algebras*, J. London Math. Soc. 26(1951), 64-71