FREE GROUPOIDS WITH $(xy)^2 = x^2y^2$

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A b s t r a c t: We investigate free objects in the variety $\mathcal{V}$ of groupoids which satisfy the law $(xy)^2 = x^2y^2$ [1]. The main results and necessary preliminary definitions are stated in Introduction. Corresponding generalizations, concerning the law $(xy)^n = x^ny^n$, are considered in the last part of the paper.

0. Introduction

First we state some necessary preliminaries.

Let $\mathbf{G} = (G, \cdot)$ be a **groupoid**, i.e. an algebra with a binary operation: $(x, y) \mapsto xy$ on G. If $a, b, c \in G$ are such that $a = bc$, then we say that b and c are **divisors** of a in $\mathbf{G}$. A sequence $a_1, a_2, \ldots$ of elements of G is a **divisor chain** in $\mathbf{G}$ if a_{i+1} is a divisor of a_i. We say that $a \in G$ is a **prime** in $\mathbf{G}$ if the set of divisors of a in $\mathbf{G}$ is empty. Thus, primes in $\mathbf{G}$ can be only the last members of divisor chains in $\mathbf{G}$.

Throughout the paper we will always write "a free groupoid" instead of "a free groupoid in the variety of all groupoids". It will be denoted by $\mathbf{F} = (F, \cdot)$, and its basis by B. (We write $\mathbf{F} = F(B)$ when it is necessary to emphasize the basis B.) It is well known (see, for example, [1], I.1) that the following properties characterize $\mathbf{F}$:

a) $ab = cd \Rightarrow a = c,\ b = d$, i.e. the mapping $(a, b) \mapsto ab$ is injective.

b) Every divisor chain in $\mathbf{F}$ is finite.

Then the set B of primes in $\mathbf{F}$ is nonempty and it is the unique basis of $\mathbf{F}$.

If $\mathbf{G} = (G, \cdot)$ is a given groupoid, then for any nonnegative integer k we define a transformation (k): $x \mapsto x^{(k)}$ of G in the following way:

$$x^{(0)} = x, \qquad x^{(k+1)} = x^{(k)}x^{(k)}. \tag{0.1}$$

From the condition a) we obtain that, in a free groupoid $\mathbf{F}$, (k) is injective, for any $k \geq 0$. Thus, for each $k \geq 0$, there exists an injective partial transformation $(-k)$: $x \mapsto x^{(-k)}$ defined in $\mathbf{F}$ as follows:

$$y^{(-k)} = x \Leftrightarrow y = x^{(k)}. \tag{0.2}$$

For any $u \in F$, there exists the largest integer $[u] = m$, such that $u^{(-m)} \in F$. (The integer $[u]$ will be called the **exponent** of u in F.)

The following subset R of F will play an important role in the paper. Namely, if B is the basis of $\mathbf{F}$, then we define R as the least subset of F such that $B \subseteq R$, and if $u = vw \in F \backslash B$, then:

$$u \in R \Leftrightarrow [v, w \in R \quad \text{and} \quad (v = w \text{ or } \min\{[v], [w]\} = 0)]. \tag{0.3}$$

Recall that we denoted by $\mathcal{V}$ the variety of groupoids which satisfy the law

$$(xy)^2 = x^2y^2. \tag{0.4}$$

If $\mathbf{G} \in \mathcal{V}$, then we call $\mathbf{G}$ a $\mathcal{V}$**–groupoid**, and if it is free in $\mathcal{V}$, we say that it is $\mathcal{V}$**–free**.

Now we are ready to state the main results.

Theorem 1. If $u, v \in R$, $m = \min\{[u], [v]\}$ and $u * v$ is defined by:

$$u * v = \left(u^{(-m)}v^{(-m)}\right)^{(m)} \tag{0.5}$$

then $\mathbf{R} = (R, *)$ is a $\mathcal{V}$–free groupoid and the set B (i.e. the basis of F) is the unique basis of $\mathbf{R}$.

Theorem 2. A $\mathcal{V}$–groupoid $\mathbf{H} = (H, \cdot)$ is $\mathcal{V}$–free iff the following conditions hold

(i) Every divisor chain in $\mathbf{H}$ is finite.

(ii) $x^2 = y^2 \Rightarrow x = y$.

(iii) $xy = uv,\ x \neq y,\ u \neq v \Rightarrow x = u,\ y = v.$ [2]

(iv) $x^2 = yz,\ y \neq z \Rightarrow (\exists u, v)\ (x = uv,\ y = u^2,\ z = v^2)$.

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[1] As usual: $x^2 = xx$.

[2] "$p, q, \ldots$" means "$p\&q\& \ldots$".

Then the set P of primes in $\mathbf{H}$ is nonempty and the unique basis of $\mathbf{H}$.

Theorem 3. If $\mathbf{H}$ is a $\mathcal{V}$–free groupoid, then there exist subgroupoids $\mathbf{G}$, $\mathbf{Q}$ of $\mathbf{H}$ such that $\mathbf{G}$ is not $\mathcal{V}$–free, and $\mathbf{Q}$ is $\mathcal{V}$–free with an infinite rank.

The next results concern a sequence of functors in the variety $\mathcal{V}$. Namely, if $\mathbf{G}$ is a groupoid and k is a nonnegative integer, then we define the groupoid $\mathbf{G}^{(k)} = (G, (k))$ as follows:

$$x\ (k)\ y = (xy)^{(k)}. \tag{0.6}$$

(Note that the same symbol (k) is used in (0.6) with two different meanings: as an operation of G on the left, and as a transformation on the right side.)

Theorem 4. If $\mathbf{H}$ is a $\mathcal{V}$–free groupoid and $k \geq 1$, then:

1) $\mathbf{H}^{(k)} \in \mathcal{V}$

2) $\mathbf{H}^{(k)}$ is not $\mathcal{V}$–free, and

3) The subgroupoid $\mathbf{Q}$ of $\mathbf{H}^{(k)}$ generated by the basis B of $\mathbf{H}$ is a $\mathcal{V}$–free groupoid with the basis B.

In §i, $1 \leq i \leq 4$, we prove *Th. i*, and in §5 we consider the law $(xy)^n = x^n y^n$, where $n \geq 3$.

1. *A canonical description of $\mathcal{V}$–free groupoids*

First we will introduce a norm of the elements of F and state some lemmas in order to prove *Th.* 1. (As was mentioned in Introduction, we denote by B the basis of a given free groupoid $\mathbf{F} = (F, \cdot)$.)

The **norm** in F is defined as the homomorphism $u \mapsto |u|$ from $\mathbf{F}$ into the additive groupoid of positive integers, which is an extension of the mapping $B \to \{1\}$. Thus:

$$|vw| = |v| + |w|\,, \qquad |b| = 1\,, \tag{1.1}$$

for all $v,\ w \in F,\ b \in B$.

In the proof of *Th.* 1 we will use some of the following relations, where $u, v \in F$ and k, m are integers.

$$u^{(k)} \in F \Leftrightarrow k + [u] \geq 0 \tag{1.2}$$

$$k + [u] \geq 0 \Rightarrow \left|u^{(k)}\right| = 2^k |u| \tag{1.3}$$

$$k + [u] \geq 0,\ k + m + [u] \geq 0 \Rightarrow \left(u^{(k)}\right)^{(m)} = u^{(k+m)} \tag{1.4}$$

$$k + [u] \geq 0,\ m - k + [v] \geq 0 \Rightarrow (u^{(k)} = v^{(m)} \Leftrightarrow u = v^{(m-k)} \tag{1.5}$$

$$k + [u] \geq 0 \Rightarrow \left(u^{(k)} \in R \Leftrightarrow u \in R\right) \tag{1.6}$$

We will also use the following two lemmas, which can be also easily shown.

Lemma 1.1. If $\varphi\colon \mathbf{Q} \to \mathbf{G}$ is a homomorphism, then

$$x \in Q, \quad m \geq 0 \Rightarrow \varphi\left(x^{(m)}\right) = \varphi(x)^{(m)}. \quad \square$$

Lemma 1.2. If $\mathbf{G} \in \mathcal{V}$ and $x,\ y \in G,\ m \geq 0$, then

$$(xy)^{(m)} = x^{(m)}\, y^{(m)}. \quad \square$$

Now we can prove *Theorem* 1.

First, if $u \in R$ is such that $[u] = m$, then by (1.4) we have

$$u * u = \left(u^{(-m)}\, u^{(-m)}\right)^{(m)} = \left(\left(u^{(-m)}\right)^{(1)}\right)^{(m)} = u^2. \tag{1.7}$$

Let $u,\ v \in R$ be such that $u \neq v$, and $\min\{[u],\ [v]\} = m$. Then $\left[u^{(-m)}\right] = 0$ or $\left[v^{(-m)}\right] = 0$, wich implies $u^{(-m)}\, v^{(-m)} \in R$, and by (1.6) we obtain $u * v \in R$. Thus:

$$u,\ v \in R \Rightarrow u * v \in R\,, \tag{1.8}$$

i.e. $\mathbf{R} = (R, *)$ is a groupoid.

Moreover, if $\min\{[u],\ [v]\} = m$, then:

$$(u * v) * (u * v) = (u * v)^2 = (u * v)^{(1)} = \left(u^{(-m)}\, v^{(-m)}\right)^{(m+1)},$$

$$(u * u) * (v * v) = u^2 * v^2 = \left((u^2)^{(-m-1)}\, (v^2)^{(-m-1)}\right)^{(m+1)} = \left(u^{(-m)}\, v^{(-m)}\right)^{(m+1)},$$

and this implies that $\mathbf{R} \in \mathcal{V}$.

If $u,\ v \in R$ are such that $uv \in R$, then $u = v$ or $\min\{[u],\ [v]\} = 0$, and thus we have:

$$u,\ v,\ uv \in R \Rightarrow u * v = uv\,, \tag{1.9}$$

and so B is a generating set of R. Clearly, B is the set of primes in $\mathbf{R}$.

By (0.5) and (1.3) we obtain:

$$|u * v| = 2^m \left|u^{(-m)}\, v^{(-m)}\right| = 2^m \left(2^{-m}|u| + 2^{-m}|v|\right) = |u| + |v|\,, \tag{1.10}$$

i.e the restriction of the norm on R is a homomorphism from $\mathbf{R}$ onto the additive groupoid of positive integers. This restriction will be called the **norm on R**.

Let $\mathbf{G} = (G, \cdot) \in \mathcal{V}$, $\lambda\colon B \to G$ be an arbitrary mapping, and $\varphi\colon \mathbf{F} \to \mathbf{G}$ be the homomorphism which is an extension of λ. Denote by ψ the restriction of φ on R. If $u, v \in R$ and $m = \min\{[u], [v]\}$, then:

$$\begin{aligned}\psi(u * v) &= \varphi\left(\left(u^{(-m)}\, v^{(-m)}\right)^{(m)}\right) = \left(\varphi\left(u^{(-m)}\, v^{(-m)}\right)\right)^{(m)} \\ &= \left(\varphi\left(u^{(-m)}\right)\varphi\left(v^{(-m)}\right)\right)^{(m)} = \left(\varphi\left(u^{(-m)}\right)\right)^{(m)}\left(\varphi\left(v^{(-m)}\right)\right)^{(m)} \\ &= \varphi\left(\left(u^{(-m)}\right)^{(m)}\right)\varphi\left(\left(v^{(-m)}\right)^{(m)}\right) = \varphi(u)\,\varphi(v) = \psi(u)\,\psi(v)\,,\end{aligned}$$

i.e. $\psi\colon \mathbf{R} \to \mathbf{G}$ is a homomorphism.

Thus $\mathbf{R}$ is a free groupoid in $\mathcal{V}$, with a basis B. B is the unique basis of $\mathbf{R}$, for it is a subset of any generating subset of $\mathbf{R}$. This completes the proof of *Th.* 1.

Remark. The above proof of *Th.* 1 is almost a direct consequence of the previously given definitions and results. Of course, some more general results could be used, but they would make the corresponding proof even more complicated. We will not include here discussions of that kind, and the interested reader is addressed to the corresponding books and papers (for example: [2], III.5; [4], §10; [5], 1.4; [6], 2.9).

2. An axiom system for $\mathcal{V}$-free groupoids

The main object of this section is the proof of **Th. 2.**

Proposition 2.1. Every $\mathcal{V}$–free groupoid satisfies the conditions (i)–(iv) of **Th. 2.**

Proof. By **Th. 1.**, it is enough to show that $\mathbf{R} = (R, *)$ satisfies (i)–(iv).

Below we assume that $x, y, z, u, v \in R$.

1) By (1.10), if $x * y = z$, then $|z| > |x|$, $|z| > |y|$, and this implies that $\mathbf{R}$ satisfies (i).

2) If $x{*}x{=}y{*}y$, then (according to (1.7)) $x^2{=}y^2$, and so $x{=}y$; thus (ii) holds.

Assume that $x * y = u * v$ and $\min\{[x], [y]\} = p \le q = \min\{[u], [v]\}$. Then, by (0.5) and (1.5):

$$x^{(-p)}\, y^{(-p)} = \left(u^{(-q)}\, v^{(-q)}\right)^{(q-p)}. \tag{2.1}$$

If $p = q$, then $x^{(-p)}\, y^{(-p)} = u^{(-p)}\, v^{(-p)}$ which implies $x = u$, $y = v$.

If $p < q$, then by (2.1):

$$x^{(-p)}\, y^{(-p)} = \left(\left(u^{(-q)}\, v^{(-q)}\right)^{(q-p-1)}\right)^2,$$

which implies $x^{(-p)} = y^{(-p)}$, i.e. $x = y$.

Thus we have:

3) $x * y = u * v,\ \ x \neq y,\ \ u \neq v \Rightarrow x = u,\ \ y = v$, i.e. (iii) is satisfied.

Finally, assume:

4) $x * x = y * z$, $y \neq z$.

Then, if $q = \min\{[y], [z]\}$, by (0.5) and (1.7), we have $x^2 = \left(y^{(-q)}\, z^{(-q)}\right)^{(q)}$, i.e.

$$x = \left(y^{(-q)}\, z^{(-q)}\right)^{(q-1)} = y^{(-1)} * z^{(-1)}.$$

Thus: $x = u * v$, $y = u^2$, $z = v^2$, where $u = y^{(-1)}, v = z^{(-1)}$. □

Now we will show the following

Lemma 2.2. Let $\mathbf{G} = (G, \cdot)$ be a groupoid which satisfies the condition (i) of *Th.* 2 and the following one:

(v) The set $\operatorname{div}(a)$ of divisors of an arbitrary element $a \in G$ is finite.

Then, for arbitrary $a \in G$, the set of lengths of divisor chains with the first member a is bounded.

(We denote by $L(a)$ the largest member of this set, and we say that $L(a)$ is the **length** of a.)

Proof. Consider the oriented graph of which the nodes are the elements of G, and for $a, b \in G$ there exists an edge with the initial node a and the terminal node b iff b is a divisor of a. From the given conditions it follows that every node of the graph is a "source" of finitely many edges and that every (directed) path in the graph is finite. Then, by König's Lemma (for example, [3,4]), one obtains that the set of path lengths, of which the origin is a given node, is bounded. □

To complete the proof of *Th.* 2 we have to show the following

Proposition 2.3. If a $\mathcal{V}$-groupoid $\mathbf{H}$ satisfies the conditions (i)–(iv), then $\mathbf{H}$ is $\mathcal{V}$-free, and the set B of primes in $\mathbf{H}$ is the basis of $\mathbf{H}$.

Proof. First, (i) implies that the set B of primes in $\mathbf{H}$ is nonempty. By (ii), (iii) and (iv), for each $a \in H$, $\mathrm{div}(a)$ consists of at most 3 elements; thus the conclusion of *L.* 2.2 holds. By induction on $L(a)$ we obtain that B is the least generating subset of $\mathbf{H}$.

Let $\mathbf{G} = (G, \cdot) \in \mathcal{V}$, and $\lambda: B \to G$ be an arbitrary mapping. Again by induction on $L(a)$ we will show that there is a (unique) homomorphism $\varphi: \mathbf{H} \to \mathbf{G}$ which is an extension of λ. First we put $\varphi(b) = \lambda(b)$ if $b \in B$. Assume that, for any $x \in H$ such that $L(x) \leq k$, $\varphi(x) \in G$ is well defined, and if $x = uv$, then $\varphi(x) = \varphi(u)\,\varphi(v)$. Let $t \in H$ be such that $L(t) = k+1$. Then t is a product, $t = uv$, where $L(u)$, $L(v) \leq k$; and, there exist at most two distinct such pairs. Then we can put $\varphi(t) = \varphi(u)\,\varphi(v)$. If $t = x^2 = yz$, where $y \neq z$, then, by (iv), there exist u, v such that: $x = uv$, $y = u^2$, $z = v^2$, and thus:

$$\varphi(x^2) = \varphi(x)\,\varphi(x) = \big(\varphi(u)\,\varphi(v)\big)^2 = \varphi(u)^2\,\varphi(v)^2 = \varphi(u^2)\,\varphi(v^2) = \varphi(y)\,\varphi(z).$$

Thus $\varphi(t) \in G$ is well defined. Moreover, we have $\varphi(uv) = \varphi(u)\,\varphi(v)$, for each u, v such that $L(uv) \leq k+1$. So, there exits a homomophism $\varphi: \mathbf{H} \to \mathbf{G}$ which is an extension of λ. □

The additive groupoid of positive integers belongs to $\mathcal{V}$, and this implies:

Proposition 2.4. If $\mathbf{H}$ is a $\mathcal{V}$-free groupoid with the basis B, then there exists a (unique) mapping $x \mapsto |x|$ from $\mathbf{H}$ into the set of positive integers such that:

$$|b| = 1, \qquad |xy| = |x| + |y|\,, \tag{2.2}$$

for each $b \in B$, x, $y \in H$. (We say that $|x|$ is the **norm** of x.) □

Now we will show the following:

Proposition 2.5. Every $\mathcal{V}$-free groupoid $\mathbf{H}$ is a cancellative groupoid.

Proof. First we will show that:

$$x \neq y \;\Rightarrow\; x^2 \neq xy\,, x^2 \neq yx\,. \tag{2.3}$$

Namely, (2.3) is clear if x, $y \in B$. Assume that x, $y \in H$ are such that $x \neq y$, $x^2 = xy$ and $|x|$ is the least possible. Then, by (iv), there exist u, $v \in H$ such that $x = uv$, $x = u^2$, $y = v^2$, $u \neq v$. Therefore $u^2 = uv$, $u \neq v$ and $|u| < |x|$.

By symmetry, $x \neq y \Rightarrow x^2 \neq yx$.

Let $xy = xz$ (or $yx = zx$). If $x \neq y$, $x \neq z$, then by (iii): $y = z$. If $x = y$, then: $x^2 = xz \Rightarrow x = z$. Thus:

$$xy = xz \quad \text{or} \quad yx = zx \;\Rightarrow\; y = z\,, \tag{2.4}$$

i.e. $\mathbf{H}$ is cancellative. □

Below we assume that $\mathbf{H}$ is a given $\mathcal{V}$-free groupoid.

As a concequence of (i) we obtain:

Corollary 2.6. For every $k \geq 0$, the mapping $x \mapsto x^{(k)}$ is injective. □

As in (0.2), the equality $x = y^{(-k)}$ is equivalent with $y = x^{(k)}$, where $k \geq 0$. Thus, for every $x \in H$, there exists the largest nonnegative integer m such that $x^{(-m)} \in H$; it will be denoted by $[x]$. Therefore, we have a mapping $x \mapsto [x]$ from H into the set of nonnegative integers. It can be easily seen that if we replace F by H in (1.2)–(1.5) we obtain relations which hold in a $\mathcal{V}$-free groupoid $\mathbf{H}$. Moreover, we obtain the following property which is an "extension" of *L.* 1.2).

Proposition 2.7. If x, $y \in H$ and m are such that $[x] + m \geq 0$, $[y] + m \geq 0$, then $[xy] + m \geq 0$ and:

$$(xy)^{(m)} = x^{(m)}\,y^{(m)}. \quad □$$

We note that:

$$x, y \in H \;\Rightarrow\; [xy] = \min\{[x], [y]\}. \tag{2.5}$$

Remark. *Th.* 2 could be stated in a weaker form, i.e. without the assumption $\mathbf{H} \in \mathcal{V}$, replacing (iv) by:

(iva) $x^2 = yz$, $y \neq z \Leftrightarrow (\exists u, v)\;\; x = uv$, $y = u^2$, $z = v^2$, $u \neq v$.

3. Subgroupoids of $\mathcal{V}$-free groupoids

Below we assume that $\mathbf{H}$ is a given $\mathcal{V}$-free groupoid with the basis B, and $\mathbf{Q}$ is a subroupoid of $\mathbf{H}$ with the carrier Q.

From *Th.* 2 one obtains:

Proposition 3.1. $\mathbf{Q} \in \mathcal{V}$ and it satisfies the conditions (i), (ii) and (iii) in *Th.* 2. □

Proposition 3.2. The set of primes in $\mathbf{Q}$ is nonempty and it is the least generating subset of $\mathbf{Q}$. □

According to L. 2.2, we have:

Proposition 3.3. If $a \in Q$, then there exists a positive integer $L_Q(a)$ such that $L_Q(a)$ is the largest length of divisor chains in $\mathbf{Q}$ with the first member a. Moreover: $L_Q(a) \leq L(a)$. □

We will show:

Proposition 3.4. If $b \in B$ and $\mathbf{Q}$ is generated by b^2, b^2b, then $\mathbf{Q}$ is not $\mathcal{V}$–free.

Proof. Clearly: b^2, $b^2b \in Q$, $(b^2)^2b^2 = (b^2b)^2 \in Q$ and $b^2 \neq b^2b$, but there is no $v \in Q$ such that $b^2 = v^2$. Thus $\mathbf{Q}$ does not satisfy (iv), i.e. $\mathbf{Q}$ is not $\mathcal{V}$–free. ($\{b^2,\ b^2b\}$ is the set of primes in $\mathbf{Q}$.) □

By Th. 2 and Pr. 3.1, we have:

Proposition 3.5. $\mathbf{Q}$ is $\mathcal{V}$–free if:

$$u \neq v,\ uv \in Q\ u^2, v^2 \in Q \Rightarrow u, v \in Q\,, \tag{3.1}$$

for any u, $v \in H$. □

As a concequence of Pr. 3.5 we have:

Propositon 3.6. Each of the following conditions is sufficient for $\mathbf{Q}$ to be $\mathcal{V}$–free:

$$x^2 \in Q \Rightarrow x \in Q\,, \tag{3.2}$$

$$u \neq v,\ uv \in Q \Rightarrow u,\ v \in Q\,. \quad \square \tag{3.3}$$

The following property will help to complete the proof of Th. 3.

Propositon 3.7. If $[u] = 0$ for every prime in $\mathbf{Q}$, then $\mathbf{Q}$ is $\mathcal{V}$–free.

Proof. It is enough to show that $\mathbf{Q}$ satisfies the condition (3.2) and this can be shown by induction on $L_Q(x^2)$. □

Propositon 3.8. Let $b \in B$, $a_1 = b^2b$, $a_{k+1} = a_kb$, $A = \{a_k \mid k \geq 1\}$. If $\mathbf{Q}$ is generated by A, then $\mathbf{Q}$ is $\mathcal{V}$–free with an infinite rank.

Proof. All the elements of A are primes in $\mathbf{Q}$, and then apply Pr. 3.7. □

Propositon 3.9. Let $C = \{b^2\} \cup A$, where A is as in Pr. 3.8. If $\mathbf{S}$ is the groupoid generated by C, then $\mathbf{S}$ is not $\mathcal{V}$–free, and all the elements of C are primes in S. □

4. Some properties of the functors (k) in $\mathcal{V}$

The proofs of the following three statements are obvious.

Propositon 4.1. $\mathbf{G} \in \mathcal{V} \Rightarrow G^{(k)} \in \mathcal{V}$. □

Propositon 4.2. If $\mathbf{G} = (G, \cdot)$, $\mathbf{S} = (S, \cdot) \in \mathcal{V}$ and $\varphi: G \to S$ is a homomorphism from $\mathbf{G}$ into $\mathbf{S}$, then $\varphi: \mathbf{G}^{(k)} \to \mathbf{S}^{(k)}$ is a homomorphism from $\mathbf{G}^{(k)}$ into $\mathbf{S}^{(k)}$ as well. □

Thus for every $k \geq 0$, (k) is a functor in $\mathcal{V}$.

Propositon 4.3. If k, $n \geq 0$ and $\mathbf{G} \in \mathcal{V}$, then $\left(\mathbf{G}^{(k)}\right)^{(n)} = \mathbf{G}^{(kn+n)}$. □

Below we assume that $\mathbf{H}$ is a $\mathcal{V}$–free groupoid, with the basis B, and that k is a positive integer. The subgroupoid of $\mathbf{H}^{(k)}$ generated by B will be denoted by $\mathbf{Q}$. Also (i), (ii), (iii) and (iv) are the conditions stated in Th. 2.

The following statements 4.4–4.6 are obvious or they can be easily shown.

Propositon 4.4. If x, y, u, $v \in H$, then: $x(k)y = u(k)v \Leftrightarrow xy = uv$. □

Propositon 4.5. If $k \geq 1$, then B is a proper subset of the set P of primes in $\mathbf{H}^{(k)}$.

(Each element $b \in B$ is prime in $\mathbf{H}^{(k)}$, and for every $u \in H$, $b \in B$, we have $ub \in P$, $ub \notin B$.) □

Propositon 4.6. $\mathbf{H}^{(k)}$ satisfies (i), (ii) and (iii) of Th. 2, but for $k \geq 1$, $\mathbf{H}^{(k)}$ is not $\mathcal{V}$–free.

(Namely, if $b \in B$, then: $b^2b\,(k)\,b^2b = (b^2)^2\,(k)\,b^2$, $(b^2)^2 \neq b^2$, but b^2 is a prime in $\mathbf{H}^{(k)}$, for $k \geq 1$. Thus $\mathbf{H}^{(k)}$ does not satisfy (iv).) □

In order to complete the proof of Th. 4, first we will show the following

Lemma 4.7. If x, y, $z \in Q$, $y \neq z$ and $x^2 = yz$, then there exist γ, $\delta \in Q$ such that $\gamma \neq \delta$ and $x = (\gamma\delta)^{(k)}$.

Proof. The equality $x^2 = yz$ implies that $[yz] \geq 1$, and (by (2.5)) we have $[y], [z] \geq 1$ and $x = (yz)^{(-1)} = y^{(-1)}\,z^{(-1)}$. Thus: $x \in Q\backslash B$, and so there exist γ, $\delta \in Q$ such that $x = \gamma(k)\delta = (\gamma\delta)^{(k)}$. It remains to show that there exist different γ, δ with the above property.

Suppose that different γ, δ with the mentioned property do not exist and put $x = \alpha_0$. Then there exists a (unique) $\alpha_1 \in Q$ such that $\alpha_0 = \alpha_1^{(k+1)}$. Let the sequence $\alpha_0, \alpha_1, \ldots, \alpha_i, \ldots$ be such that $\alpha_i = \alpha_{i+1}^{(k+1)}$, for every i. Since $|\alpha_i| > |\alpha_{i+1}|$, the sequence is finite. Let α_n be its last member. By the definition of the sequence we have:

$$\alpha_0 = y^{(-1)} z^{(-1)}, \qquad \alpha_1 = y^{(-k-2)} z^{(-k-2)},$$
$$\alpha_2 = y^{(-2k-3)} z^{(-2k-3)}, \ldots, \quad \alpha_n = y^{(-nk-n-1)} z^{(-nk-n-1)}.$$

By the last equlity, there exist $u, v \in Q$ such that $\alpha_n = u(k)v = (uv)^{(k)}$. Clearly, $u \neq v$ (since α_n is the last member of the sequence). From the equality $(uv)^{(k)} = y^{(-nk-n-1)} z^{(-nk-n-1)}$ we have:

$$u = y^{(-nk-n-k-1)}, \qquad v = z^{(-nk-n-k-1)}, \qquad \text{i.e.}$$
$$y = u^{(nk+n+k+1)}, \qquad z = v^{(nk+n+k+1)}.$$

Therefore:

$$x = \alpha_0 = y^{(-1)} z^{(-1)} = u^{(nk+n+k)} v^{(nk+n+k)} =$$
$$= \left(u^{(nk+n)} v^{(nk+n)}\right)^{(k)} = (\gamma\delta)^{(k)},$$

where $\gamma = u^{(nk+n)} \neq v^{(nk+n)} = \delta$. □

Propositon 4.8. $\mathbf{Q}$ is $\mathcal{V}$–free.

Proof. Let $x, y, z \in Q$ be such that $x^2 = uz$, $y \neq z$. By *L.* 4.7, there exist $\gamma, \delta \in Q$ such that $\gamma \neq \delta$, $x = (\gamma\delta)^{(k)}$. By $x^2 = yz$ it follows that $\gamma^{(k+1)} \delta^{(k+1)} = yz$, i.e $y = \gamma^{(k+1)}$, $z = \delta^{(k+1)}$. Therefore: $x = \gamma(k)\delta$, $y = \gamma(k)\gamma$, $z = \delta(k)\delta$. Thus $\mathbf{Q}$ satisfies the condition (iv), and from *Pr.* 4.6 it follows that $\mathbf{Q}$ satisfies (i), (ii), (iii) as well. □

5. *On the equation* $(xy)^n = x^n y^n$

Denote by $\mathcal{V}_n$ the variety of groupoids which satisfy the identity

$$(xy)^n = x^n y^n, \tag{5.1}$$

where n is a given positive integer. Here the powers are defined in the usual way, i.e. by:

$$x^1 = x, \qquad x^{k+1} = x^k x. \tag{5.2}$$

By the above considerations, $\mathcal{V}_1$ is the variety of all groupoids, and $\mathcal{V}_2$ the variety $\mathcal{V}$. Further on we assume that n is a fixed integer and $n \geq 2$.

Define $x^{(k)}$, for $k \geq 0$, by:

$$x^{(0)} = x, \qquad x^{(k+1)} = \left(x^{(k)}\right)^n. \tag{5.3}$$

Note that (5.3), for $n = 2$, coincides with (0.1).

As before, $\mathbf{F} = (F, \cdot)$ denotes a free groupoid with the basis B. Since the implication

$$x^k = y^m \;\Rightarrow\; x = y, \quad k = m \tag{5.4}$$

is true in $\mathbf{F}$, the mapping $x \mapsto x^{(m)}$ is an injective transformation of F. Thus we can define $x^{(-k)}$ and $[x]$, as in the special case $n = 2$.

It is easy to show that (1.2)–(1.6), *L.* 1.1 and *L.* 1.2 are true for any $n \geq 2$.

Now we will define F_n as the least subset of F such that $B \subseteq F_n$ and:

$$vw \in F_n \;\Leftrightarrow\; \left[(w \in F_n, v = w^{n-1}) \text{ or } (v, w \in F_n \text{ and } \min\{[v],[w]\} = 0)\right]. \tag{5.5}$$

Therefore, $F_2 = R$ where R is defined by (0.3). Note that the implication $vw \in F_n \Rightarrow v, w \in F_n$, for $n \geq 3$, is not true. (For example, if $b \in B$ and $n = 3$, then $b^{(2)} = \left(b^3\right)^2 \cdot b^3 \in F_3$, but $\left(b^3\right)^2 \notin F_3$.)

The following statement is a generalization of *Th.* 1.

Theorem 1′. $\mathbf{F}_n = (F_n, *)$ is a $\mathcal{V}_n$–free groupoid with the unique basis B. Here:

$$u * v = \left(u^{(-m)} v^{(-m)}\right)^{(m)},$$

where $u, v \in F_n$ and $m = \min\{[u], [v]\}$. □.

This generalization is obtained by substituing R by F_n. The situation with the other theorems is similar, except with *Th.* 4. Namely, the definition of the operation (k), given by (0.6), does make sense for $n \geq 3$ also, but it is easy to show that $\mathbf{F}_n^{(k)} \notin \mathcal{V}_n$, for $n \geq 3$.

The statements (ii), (iii) and (iv) of *Th.* 2, in the formulation of *Th.* 2′ (besides the substitution of $\mathcal{V}$ by $\mathcal{V}_n$), obtain the following forms:

(ii′) $x^n = y^n \Rightarrow x = y$.

(iii′) $xy = uv$, $x \neq y^{n-1}$, $u \neq v^{n-1} \Rightarrow x = u$, $y = v$.

(iv′) $x^n = yz$, $y \neq z^{n-1} \Rightarrow (\exists u, v) x = uv$, $y = u^n$, $z = v^n$.

According to *Th.* 3′, note that if $\mathbf{H}$ is a $\mathcal{V}_n$–free groupoid and if $\mathbf{Q}$ is the subgroupoid of $\mathbf{H}$ generated by $A = \{a_p | p \geq 1\}$, where $a_p = b^{n+p}$ (b is an element of the basis B), then A is the basis of $\mathbf{Q}$. Therefore, $\mathbf{Q}$ has an infinite rank. If $\mathbf{S}_p$

is generated by $\{b^n, a_p\}$, then $\mathbf{S}_p$ is not $\mathcal{V}_n$–free, and $\{b_n, a_p\}$ is the set of primes in $\mathbf{S}_p$. The groupoid $\mathbf{S}$ generated by $C = \{b^n\} \cup A$ is not $\mathcal{V}_n$–free, and C is the set of primes in $\mathbf{S}$.

For a given positive integer n, there are $(2n-2)!/n!(n-1)! = A_n$ different possibilities of defining n–th powers, i.e. transformations $x \mapsto x^n$ in groupoids (see, for example [2], III.2, or [7], I.4). Therefore, there exist A_n varieties of groupoids each of which is defined by an equality of the form (5.1).

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Резиме

СЛОБОДНИ ГРУПОИДИ СО РАВЕНСТВО $(xy)^2 = x^2y^2$

Се разгледуваат слободните објекти во многуобразието групоиди што го задоволуваат равенството $(xy)^2 = x^2y^2$. Главните резултати се формулираат во воведот. Во наредните четири раздели се испитуваат својствата на слободните објекти и се даваат докази на резултатите формулирани во воведот. Во последниот раздел се разгледува поопштиот случај на групоиди што го задоволуваат равенството $(xy)^n = x^ny^n$.