SOME APPLICATIONS OF GROUPS IN INVESTIGATIONS OF (*n*, *m*)-GROUPS

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A b s t r a c t: In several papers (e.g. [1] and [3]) the usefulness of the Post Coset Theorem for investigating polyadic groups is pointed out. The aim of this paper is to illustrate the method of using binary groups for investigating vector valued groups, refering to the main results in [11].

In [3] the method of proving theorems on n-groups using the Post Coset Theorem is called an "indirect method". We also refer to a method of proving theorems for (n, m)-groups as an "indirect one" if it uses binary groups. In [5, Cor 8.4] the following result is proved:

"*An* (n, m)*-semigroup* (Q, f) *is an* (n, m)*-group iff the condition* 4) *in Theorem* 1 (in this paper) *holds*."

This result will be used below to illustrate the usefulness of the "indirect method".

However, many authors prefer the "direct method" of proving (i.e. without using Post Coset Theorem or its consequences). This also refers to professor Ušan, one of the most prolific current researchers on n-groups. (He is, to our knowledge, the first one who gave a characterization of n-groups as algebras of type $\langle n, n-1, n-2\rangle$ ([9]) which generalizes the well known result on groups as algebras of type $\langle 2, 1, 0\rangle$.) In his paper ([11]), which appeared several months ago, an axiom system for (n, m)-groups is obtained when $2m \le n < 3m$ (Theorem 3.1), and $n \ge 3m$ (Theorem 3.2), which is a generalization of the axiom system of [9]. Each of these axiom systems consists of four equations: in two of them a corresponding partial associativity of the "main operation" f is assumed and the other two equations connect the operation f with the operations e and ι, called by the author *neutral* and *inverse* one, respectively.

In this paper, using the condition 4) in Theorem 1, and supposing that (Q, f) is an (n, m)-semigroup (where n, m are any positive integers such that $n - m = k \ge 1$), we prove that (Q, f) is an (n, m)-group iff for some positive integer s, such that $sk \ge m$, there are operations e_s, ι_s which satisfy the corresponding equations (6), below. Therefore, by 4') of Theorem 1, if $k \ge m$, one can choose $s = 1$, and then e_1, ι_1 are the neutral and inverse operations e, $^{-1}$, respectively, mentioned in Theorem 3.1 and Theorem 3.2 of [11]; for $k < m$, each s with the above property is strictly greater than 1. The existence of neutral and inverse operations is shown in [11] only for $k \ge m$. In this paper we show that there are corresponding neutral and inverse operations e_s and ι_s for $k < m$, as well, but the least s in this case is greater than 1.

Now we state some necessary preliminaries.

Assuming that Q is a nonempty set, we denote by Q^i the i-th Cartesian power of Q, and $Q^* = \cup\{Q^i | i \ge 0\}$, $Q^+ = Q^* \setminus Q^0$. The elements of Q^* will be denoted by $\boldsymbol{a}, \boldsymbol{b}, \ldots, \boldsymbol{x}, \ldots$, and $|\boldsymbol{a}| = i$ will mean that $\boldsymbol{a} \in Q^i$; we will also write a_1^i instead of $\boldsymbol{a} = (a_1, \ldots, a_i)$, where $a_\nu \in Q$. Considering that Q^* (Q^+) is a free monoid (semigroup) with the basis Q, the equation $\boldsymbol{ab}\ldots\boldsymbol{c} = \boldsymbol{d}$, where $\boldsymbol{a}, \boldsymbol{b}, \ldots, \boldsymbol{c}, \boldsymbol{d} \in Q^*$, will have the usual meaning.

We also assume below that n, m, k are given positive integers such that $n = m + k$.

If f is a mapping from Q^n into Q^m, then the pair (Q, f) is called an (n, m)*-groupoid*. If, moreover, f is associative, i.e. the following equations

$$f(f(\boldsymbol{x})\boldsymbol{y}) = f(\boldsymbol{t}\, f(\boldsymbol{u})\boldsymbol{v}) \tag{1}$$

hold, for any $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}, \boldsymbol{u}, \boldsymbol{v} \in Q^*$, such that $|\boldsymbol{x}| = n = |\boldsymbol{u}|$ and $\boldsymbol{xy} = \boldsymbol{tuv}$, $|\boldsymbol{y}| = |\boldsymbol{tv}| = k$, then (Q, f) is called an (n, m)*-semigroup*.

To any (n,m)-semigroup $\boldsymbol{Q} = (Q, f)$, a semigroup $\boldsymbol{Q}^\wedge = (Q^\wedge, \cdot)$ is associated as follows.

Let $\approx$ be the least congruence on Q^+ such that

$$f(\boldsymbol{a}) = \boldsymbol{b} \Rightarrow \boldsymbol{a} \approx \boldsymbol{b} \tag{2}$$

Then the quotient semigroup $Q^+_{/\approx}$ is called a *universal covering* of $\boldsymbol{Q}$ and is denoted by $(Q^\wedge, \cdot)$.

Proposition 1. *Let* (Q, f) *be an* (n, m)*-semigroup,* $\tau : Q^+ \to Q^\wedge$ *be the canonical homomorphism, and* $Q_i = \tau(Q^i)$. *Then* [5; Th. 7.3., Pr. 7.4., page 55]:

a) $Q^\wedge = \cup\{Q_i \mid 1 \le i \le m + k-1\}$.

b) *The restrictions of* τ *on* Q *and* Q^m *are injections.*

c)} $Q_{m+i} \cap Q_{m+j} = \varnothing$, *for any* $0 \le i < j \le k-1$.

d) $Q^\vee = \cup \{Q_{m+i} \mid 0 \le i < k\}$ *is an ideal of* $Q^\wedge$.

e) If $m \ge 2$, then $\boldsymbol{Q}^\wedge$ *is not a group.* □

For any $f: Q^n \to Q^m$ we define $f^s : Q^{m+sk} \to Q^m$ by: $f^1 = f$ and

$$f^{s+1}(\boldsymbol{xy}) = f(f^s(\boldsymbol{x})\boldsymbol{y}), \qquad (3)$$

where $|\boldsymbol{x}| = m + sk$, $|\boldsymbol{y}| = k$. By induction one obtains that, if (Q, f) is an (n, m)-semigroup, then

$$f^s(\boldsymbol{x}\, f^r(\boldsymbol{y})\boldsymbol{z}) = f^{r+s}(\boldsymbol{xyz}), \qquad (4)$$

for any positive integers r, s and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in Q^\wedge$ such that $|\boldsymbol{y}| = m + rk$, $|\boldsymbol{xz}| = sk$.

As a corollary one obtains:

Proposition 2. *If* (Q, f) *is an* $(m + k, m)$*-semigroup, then* (Q, f^s) *is an* $(m + sk, m)$*-semigroup, as well* (see [5; Pr. 5.5]). □

An (n, m)-semigroup (Q, f) is called an (n, m)*-group* if the following condition holds:

$$(\forall \boldsymbol{a} \in Q^k, \boldsymbol{b} \in Q^m)(\exists \boldsymbol{x}, \boldsymbol{y} \in Q^m)\, f(\boldsymbol{ax}) = \boldsymbol{b} = f(\boldsymbol{ya}). \qquad (5)$$

Next Theorem gives some characterizations of (n, m)-groups.

Theorem 1. *If* (Q, f) *is an* (n, m)*-semigroup, then the following conditions are equivalent*:

1) (Q, f) *is an* (n, m)*-group.*

2) $Q^\vee$ *is a subgroup of* $Q^\wedge$.

3) *For some positive integer* s, (Q, f^s) is an $(m + sk, m)$*-group.*

4) *For some positive integer* s *such that* $sk \ge m$, *and any* $\boldsymbol{a} \in Q^{sk-m}$, *the groupoid* $(Q^m, @)$ *defined by* $(\forall \boldsymbol{x}, \boldsymbol{y} \in Q^m)\boldsymbol{x} @ \boldsymbol{y} = f^s(\boldsymbol{xay})$, *is a group.*

5) *For some positive integer* s *such that* $sk \ge m$, *there are mappings* $e : Q^{sk-m} \to Q^m$, $\iota : Q^{sk} \to Q^m$ *such that*:

$$(\forall \boldsymbol{a} \in Q^{sk-m}, \boldsymbol{x} \in Q^m)[f^s(\boldsymbol{xa}\, e(\boldsymbol{a})) = \boldsymbol{x} \,\&\, f^s(\boldsymbol{xa}\, \iota(\boldsymbol{ax})) = e(\boldsymbol{a})]. \qquad (6)$$

Moreover, the conditions 3'), 4') *and* 5'), *which would be obtained by the replacement of "some" by "any" in* 3), 4) *and* 5) *respectively, hold as well.*

Proof. In [5; Th 5.8, Cor 8.1, 8.4] it is shown that the conditions 1), 2), 3), 4), 3') and 4') are equivalent. To show that 5) is equivalent to any of these conditions, we will prove that 1) $\Rightarrow$ 5) and 5) $\Rightarrow$ 4).

Assume that (Q, f) is an (n, m)-group and s a positive integer such that $sk \ge m$. Let $\boldsymbol{a} \in Q^{sk-m}$, $\boldsymbol{u} \in Q^m$. By 4'), $(Q^m, @)$ is a group. Let $e(\boldsymbol{a})$ (respectively $\iota(\boldsymbol{au})$) be the neutral element (the inverse of $\boldsymbol{u}$) in $(Q^m, @)$. Then (6) holds. Thus 1) $\Rightarrow$ 5).

Assume now that there is a positive integer s such that $sk \ge m$, and there are mappings e: $Q^{sk-m} \to Q^m$, $\iota : Q^{sk} \to Q^m$ which satisfy (6). If $\boldsymbol{a} \in Q^{sk-m}$, $\boldsymbol{u} \in Q^m$, then by (6), $e(\boldsymbol{a})$ is a right neutral element, and $\iota(\boldsymbol{au})$ a right invers of $\boldsymbol{u} \in (Q^m, @)$; therefore $(Q^m, @)$ is a group. Thus 5) $\Rightarrow$ 4). □

We note that neutral operations for n-groupoids ((n, m)-groupoids) are introduced in [7] and [8], respectively; inverse operation for n-groups is introduced in [10], and for (n, m)-groups in [11]. Theorem 1 implies that each of them depends on s, and, for a given s, they are unique. In other words, the sets $\{e_s | s \ge 1, sk \ge m\}$ of neutral operations, and $\{\iota_s | s \ge 1, sk \ge m\}$ of inverse operations in an (n, m)-group (Q, f) are infinite. Moreover, from the proof of Theorem 1 it follows that, if $\boldsymbol{a} \in Q^{sk-m}$ and $\boldsymbol{x} \in Q^m$ are given, then $e(\boldsymbol{a})$ is the identity in the group $(Q^m, @)$, and $\iota(\boldsymbol{ax})$ is the inverse of $\boldsymbol{x}$ in the same group.

It seems that the group $Q^\vee$, called a *universal covering group* of the (n, m)-group (Q, f), is more convenient for obtaining more suitable expressions of $e(\boldsymbol{a})$ and $\iota(\boldsymbol{ax})$. For that purpose, in the Proposition that follows, we state some properties of $Q^\vee$ (see [5; §8]), where p is the least nonnegative integer, such that $m + p \equiv 0 (\mathrm{mod}\ k)$.

Proposition 3. *If* (Q, f) *is an* (n,m)*-group and* $(Q^\vee, \cdot)$ *its universal covering group, then the following statements are satisfied.*

a) *Each* $x \in Q^{\vee}$ *is a product* $x = x_1 \cdot \ldots \cdot x_{m+i}$, *where* $x_\nu \in Q$, $0 \le i < k$.

b) *If* $x_1 \cdot \ldots \cdot x_m = y_1 \cdot \ldots \cdot y_m$, $x_\nu, y_\nu \in Q$, *then* $x_1 = y_1, \ldots, x_m = y_m$.

c) *If* $\mathbf{1}$ *is the identity in* $Q^{\vee}$, *then* $\mathbf{1} = x_1 \cdot \ldots \cdot x_{m+p}$, *for some* $x_1, \ldots, x_{m+p} \in Q$.

d) *Let* $x \in Q$. *Then*

d.1) $x^{-1} = x_1 \cdot \ldots \cdot x_{m+p-1}$, *for some* $x_1, \ldots, x_{m+p-1} \in Q$, *if* $p \ge 1$,

d.2) $x^{-1} = x_1 \cdot \ldots \cdot x_{m+k-1}$, *for some* $x_1, \ldots, x_{m+k-1} \in Q$, *if* $p = 0$.

e) *If* $x_1, \ldots, x_i, y_1, \ldots, y_j \in Q$, then

$x_1 \cdot \ldots \cdot x_i = y_1 \cdot \ldots \cdot y_j \Leftrightarrow (\exists z_1, \ldots z_q)[f^r(x_1^{i}, z_1^{q}) = f^s(y_1^{j}, z_1^{q})]$. □

As a corollary, it follows that

$$e(\boldsymbol{a}) = (a_1 \ldots a_{sk-m})^{-1},\ \iota(\boldsymbol{ax}) = (a_1 \ldots a_{sk-m} x_1 \ldots x_m a_1 \ldots a_{sk-m})^{-1}, \qquad (7)$$

for any positive integer s, such that $sk \ge m$, $\boldsymbol{a} = a_1^{sk-m} \in Q^{sk-m}$, $\boldsymbol{x} = x_1^{m} \in Q^{m}$.

(Clearly, (7) could be written in the following way:

$$e(\boldsymbol{a}) = \boldsymbol{a}^{-1},\ \iota(\boldsymbol{ax}) = (\boldsymbol{axa})^{-1}, \qquad (7')$$

but then $\boldsymbol{a}$ and $\boldsymbol{x}$ in the right-hand side would be "products" in $Q^{\vee}$, and elements of Q^{+}, in the left-hand one.)

The equality $sk - m = 0$ is possible only if $k|m$, and then $p = 0$, i.e. there are unique $y_1, \ldots, y_m \in Q$ such that $\mathbf{1} = y_1 \cdot \ldots \cdot y_m$. Then we have:

$$e = (y_1 \cdot \ldots \cdot y_m)^{-1},\ \iota(\boldsymbol{x}) = (x_1 \cdot \ldots \cdot x_m)^{-1}. \qquad (8)$$

In the case when $m = 1$, the $(n, 1)$-group (Q, f) is an n-group, $k = n - 1$ and thus $Q^{\wedge} = Q^{\vee}$. The equations (7) now have the following form:

$$e(\boldsymbol{a}) = (a_1 \cdot \ldots \cdot a_{n-2})^{-1},\ \iota(\boldsymbol{ax}) = (a_1 \cdot \ldots \cdot a_{n-2} \cdot x \cdot a_1 \cdot \ldots \cdot a_{n-2})^{-1}. \qquad (7'')$$

Remark 1. Since in Theorem 3.1 and Theorem 3.2 of [11] only partial associativity of the operation f is assumed, these theorems are not corollaries of Theorem 1 of this paper. However, Theorem 1 could be used to unite both Theorem 3.1 and Theorem 3.2 and after the associativity of f has been established, the conclusion would be a consequence of Theorem 1. Theorem 1 covers any relation between the positive integers k and m, while in [11] only the case $k \ge m$ is solved. Thus, the question which axioms for partial associativity are sufficient in the case $k < m$ remains open.

We note that any commutative n-groupoid which satisfies a corresponding partial associativity is an n-semigroup. The same is true for the fully commutative (n, m)-groupoids ([6. p.7]). Thus, in this case, any partial associativity of f and the requirement for existence of operations e_s and ι_s, under the condition $sk \ge m$ can be taken as an axiom system for a fully commutative (n,m)-group (Q, f).

Remark 2. The equations (7") could be used to determine neutral and inverse operation in the additive n-group of an (n, m)-ring $(R; f, g)$. First recall that $(R; f, g)$ is an *(n, m)-ring* if (R, f) is an Abelian n-group, and g an m-operation on R distributive with respect to f. In this case, the universal covering group $R^{\wedge}$ of the n-group (R, f) is commutative, and, with the additive notation, we have:

$$R^{\wedge} = R \cup 2R \cup \ldots \cup (n-1)R.$$

Then there is a unique extension $g^{\wedge}$ of g such that $(R^{\wedge}, +, g^{\wedge})$ is a $(2, m)$-ring ([2, p.7]).

According to the additive notation, the relations (7") obtain the following forms:

$$e(a_1, \ldots, a_{n-2}) = -(a_1 + \ldots + a_{n-2}),$$

$$\iota(a_1, \ldots, a_{n-2}, x) = -(a_1 + \ldots + a_{n-2} + x + a_1 + \ldots + a_{n-2}) =$$

$$= -(2a_1 + \ldots + 2a_{n-2} + x).$$

These relations imply distributivity of g with respect to e and ι ([12]).

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Резиме

НЕКОИ ПРИМЕНИ НА ГРУПИ ЗА ИСПИТУВАЊЕ НА (*n, m*)-ГРУПИ

Ползата од Постовата теорема за испитување на полиадични групи е истакната во неколку работи, на пример [1] и [3]. Целта на оваа работа е да се илустрира методот на користење бинарни групи за испитување векторско вредносни групи, имајќи ги предвид главните резултати во [11].