ON FULLY COMMUTATIVE (*n*,*m*)-GROUPOIDS

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A b s t r a c t: In several papers (for ex. [2] and [8]) on fully commutative vector valued groupoids, corresponding properties by means of commutative vector valued groupoids are given. The goal of this paper is to give a direct investigation of the class of fully commutative vector valued groupoids without leaving the very class.

1. PRELIMINARIES

In this section we introduce some notations and define some notions used in this paper. We also give a short summary of sections 2, 3 and 4.

1.1. Throughout this paper, $Q^{(+)}$ denotes a free commutative semigroup with a given basis Q, where the operation is denoted multiplicatively, i.e., the operation sign is omitted.

Below we give a list of notations, notions, and properties used later on in the paper:

1) The mapping $x \mapsto |x|$ is the homomorphism from $Q^{(+)}$ into the additive semigroup of positive integers, such that $|b| = 1$ for each $b \in Q$.

2) If r is a positive integer, then

$$Q^{(r)} = \{x \in Q^{(+)};\quad |x| = r\}.$$

3) For a given $a \in Q$, $x \mapsto |x|$, is the homomorphism from $Q^{(+)}$ into the additive semigroup of non-negative integers, such that $|a|_a = 1$, $|b|_a = 0$, for $b \neq a$.

4) For $x \in Q^{(+)}$ the set $cn(x) = \{b \in Q: \ |x|_b > 0\}$ is the *content* of *x*.

5) If $x, y \in Q^{(+)}$, then

$$\rho(x, y) = \sum_{a \in Q} \big|\, |x|_a - |y|_a \big|$$

is called the *distance between x and y*.

From the given definitions it follows that:

6) The semigroup $Q^{(+)}$ is cancellative.

7) For each $x \in Q^{(+)}$, $|x| = \sum_{a \in Q} |x|_a = \sum_{a \in cn(x)} |x|_a$ is a positive integer.

8) For each pair $x, y \in Q^{(+)}$, the distance $\rho(x, y)$ is a non-negative integer, where $\rho(x, y) = 0$ iff $x = y$.

For technical reasons, we will add to $Q^{(+)}$ an exterior unit e (i.e $e \notin Q^{(+)}$) and obtain that $Q^{(*)} = Q^{(+)} \cup \{e\}$ is a free commutative monoid with a unit element and the basis Q. Therefore e has the following properties:

9) $e \cdot x = x = x \cdot e$, for every $x \in Q^{(*)}$; $|e| = 0$, $cn(e) = \emptyset$, $Q^{(0)} = \{e\}$, $x^0 = e$, $\rho(e, x) = |x|$, for each $x \in Q^{(*)}$.

1.2. Further on we assume that (n, m) is a pair of positive integers, such that $n - m = k \geq 1$.

If $f: x \mapsto f(x)$ is a mapping from $Q^{(n)}$ into $Q^{(m)}$, then we call f a *fully commutative (n, m)-operation* and the pair $\mathbf{Q} = (Q, f)$ a *fully commutative (n, m)-groupoid*. This notion is introduced first in [3] and [2], but it seems necessary to explain the prefix "fully commutative". One of the reasons is the fact that the term "commutative (*n*, *m*)-groupoid" is used in [1] and [4] for a pair of the form (Q, f), such that f is a mapping from Q^n into Q^m with a corresponding property of commutativity.

In this paper we will consider only *fully commutative (n, m)-groupoids*, and thus we will usually omit the prefix "fully commutative".

1.3. In section 2 we define an infinite set $\{f^r : r \geq 2\}$ of polynomial mappings in an (n, m)-groupoid (Q, f), and here we will concentrate only to f^2. Namely, f^2 is a mapping from $Q^{(n)} \times Q^{(k)}$ into $Q^{(m)}$ defined by

$$f^2(x, y) = f(f(x)y), \text{ for } x \in Q^{(n)}, y \in Q^{(k)}.$$

Using this, it follows that there is at most one $(n + k, m)$-operation $f^{(2)}$, such that $f^{(2)}(xy) = f^2(x, y)$. It comes out that for each pair (n, m) and a set Q with at least two distinct elements, there is an (n, m)-groupoid (Q, f) which does not allow $(n + k, m)$-operation $f^{(2)}$ with the mentioned property. However, if such an $(n + k, m)$-operation exists, then (Q, f) is said to be a *fully commutative (n, m)-semigroup*. This definition of (n, m)-semigroup is equivalent to the definition in [2].

It is assumed in section 3 that (Q, f) is a given $(m + k, m)$-semigroup and the General Associative Law is proved. We note that this result is also proved in [2]. It took place in this paper, too, because its proof is such that we do not leave the class of fully commutative (n, m)-groupoids.

2. POLYNOMIAL MAPPINGS IN AN (n, m)-GROUPOID

First we will prove the following:

Proposition 2.1. *If Q has at least two distinct elements, then there is an (n, m)-groupoid (Q, f)* such that

$$x\, y = x'y' \ \ \&\ \ f(f(x)\, y) \neq f(f(x')\, y'), \tag{2.1}$$

for some $x,\ x' \in Q^{(n)},\ \ y, y' \in Q^{(k)}$.

Proof. Let $a,\ b \in Q,\ a \neq b$ and let $x, x' \in Q^{(n)},\ \ y, y' \in Q^{(k)}$ be defined as follows:

$$x = a^n, \quad y = b^k, \quad x' = a^{n-1}b, \quad y' = ab^{k-1}.$$

If f is a mapping from $Q^{(n)}$ into $Q^{(m)}$ with the properties

$$a^m = f(a^n) = f(a^m b^k),\ b^m = f(a^{n-1}b) = f(ab^{n-1}),$$

then $x, x' \in Q^{(n)}$ and $y, y' \in Q^{(k)}$ are such that (2.1) holds. □

Below we assume that (Q, f) is an (n, m)-groupoid.

We define a set of mappings $\{f^r : r \geq 1\}$ as follows. First, $f^1 = f$. Assuming that $f^r : D_r \to Q^{(m)}$ is defined, we define $f^{r+1} : D_{r+1} \to Q^{(m)}$, where $D_{r+1} = D_r \times Q^{(k)}$ $(D_1 = Q^{(n)})$, by

$$f^{r+1}(x, y) = f(f^r(x)y), \tag{2.2}$$

for $x \in D_r,\ \ y \in Q^{(k)}$.

Since $D_1 = Q^{(n)}$, one obtains that

$$D_r = Q^{(n)} \times (Q^{(k)})^{r-1} = \{(x_1, x_2, \ldots, x_r) : x_1 \in Q^{(n)},\ x_2, \ldots, x_r \in Q^{(k)}\},$$

for each $r \geq 2$.

The following proposition will be used in section 3.

Proposition 2.2. *For each pair $(p,\ q)$ of positive integers and $x_1 \in Q^{(n)},\ \ x_2, \ldots, x_{p+q} \in Q^{(k)}$, it holds that*

$$f^{p+q}(x_1, x_2, \ldots, x_{p+q}) = f^p(f^q(x_1, \ldots, x_q)x_{q+1}, \ldots, x_{p+q}). \tag{2.3}$$

Proof. For $p = 1$, (2.3) holds by (2.2). For $q = 1$, (2.3) can be shown by induction on p, and for $p \geq 2,\ \ q \geq 2$ by induction on $p + q$. □

The following statement is clear.

Proposition 2.3. *If $r \geq 2$, then there is at most one $(m + rk,\ m)$-operation $f^{(r)}$, such that*

$$f^{(r)}(x_1 x_2 \ldots x_r) = f^r(x_1, x_2, \ldots, x_r) \tag{2.4}$$

for each $(x_1, x_2, \ldots, x_r) \in D_r$. Such an operation $f^{(r)}$ do exist iff the equality:

$$f^r(x_1, x_2, \ldots, x_r) = f^r(x'_1, x'_2, \ldots, x'_r), \tag{2.5}$$

holds for every $(x_1, x_2, \ldots, x_r), (x'_1, x'_2, \ldots, x'_r) \in D_r$, such that $x_1 x_2 \ldots x_r = x'_1 x'_2 \ldots x'_r$. □

We say that (Q, f) is a fully commutative (n, m)-semigroup iff $(Q, f^{(2)})$ is a well-defined fully commutative $(n + k,\ m)$-groupoid. As is mentioned in 1.2, we will omit the prefix "fully commutative" and say simply "(n, m)-semigroup". By Proposition 2.3, it follows that (Q, f) is an (n, m)-semigroup iff the equality

$$f^2(x, y) = f^2(x', y'), \tag{2.6}$$

i.e.,

$$f(f(x)y) = f(f(x')y'), \tag{2.7}$$

holds for all $x, x' \in Q^{(n)},\ \ y, y' \in Q^{(k)}$, such that $xy = x'y'$.

By the proposition that will be proved below, it follows that this definition of the notion of an (n, m)-semigroup is equivalent to the definition in [2].

Proposition 2.4. *The following conditions are equivalent:*

(i) (Q, f) is an (n, m)-semigroup.

(ii) The equality

$$f(f(xa)by) = f(f(xb)ay) \tag{2.8}$$

holds, for every $a, b \in Q,\ \ x \in Q^{(n-1)},\ \ y \in Q^{(k-1)}$.

Proof. $(i) \Rightarrow (ii)$ follows by (2.7). Suppose that (ii) holds, but that (i) does not. Then there are $t, t' \in Q^{(n)}$ and $u, u' \in Q^{(k)}$, such that

$$tu = t'u',\quad f(f(t)u) \neq f(f(t')u'), \tag{2.9}$$

where $\rho(u, u')$ is the lowest possible value. Then $u \neq u'$, because for $u = u'$ we would have $t = t'$, which contradicts the inequality in (2.9). Therefore

$s=\rho(u,u')>0$. Since $|u|=|u'|$, one obtains that there are $a,b\in Q$, such that $|u|_a>|u'|_a$, $|u|_b<|u'|_b$. Then there are $x\in Q^{(n-1)}$, $y\in Q^{(k-1)}$, such that $u=ay$, $t=xb$. By (*ii*):

$$f(f(t)u)=f(f(xb)ay)=f(f(xa)by),\quad r=\rho\,(by,u')<s=\rho\,(u,u'),$$

which is a contradiction of the minimality of s. □

Remark. Supposing that $k\geq m$, beside the infinite set of polynomial mappings $\{f^r:r\geq 1\}$ in the given (n,m)-groupoid (Q,f), one can also define a finite set of polynomials as follows.

First, since $k\geq m$, there is a unique pair of integers (q,p) such that $k=qm+p$, $q\geq 1$, $0\leq p<m$. Therefore, if $x_1,x_2,\dots,x_r\in Q^{(n)}$, $y\in Q^{(s)}$ are such that $rm+s=n\ (=(q+1)m+p)$, then

$$z=f(f(x_1)f(x_2)\dots f(x_r)y)\in Q^{(m)} \tag{2.10}$$

and z does not depend on the disposition of $x_1,x_2,\dots,x_r$. For $r=1$, we have $f^2(x_1,y)$, and hence we suppose that $r\geq 2$. Then, $s=(q+1-r)m+p$.

Let $(Q^{(n)})^{(+)}$ be a free commutative semigroup with the basis $Q^{(n)}$, where the product of $x_1,x_2,\dots,x_r\in Q^{(n)}$ will be denoted by $x_1\bullet x_2\bullet\dots\bullet x_r$ in $(Q^{(n)})^{(+)}$ to make a distinction of their product $x_1x_2\dots x_r$ in $Q^{(+)}$. If we put $D_{[r]}=(Q^{(n)})^{(r)}\times Q^{(s)}$ and

$$f^{[r+1]}(x_1\bullet x_2\bullet\dots\bullet x_r,y)=f(f(x_1)f(x_2)\dots f(x_r)y), \tag{2.11}$$

we come to a new set $\{f^{[3]},f^{[4]},\dots,f^{[q+2]}\}$ of polynomial operations, such that $f^{[r+1]}$ has the domain $D_{[r]}$ and is defined by (2.11). Also, for $r=q+1$ in (2.11) we have that $s=p$. Hence, for $p=0$ and $r=q+1$ we have that $D_{[r]}=(Q^{(n)})^{(d+1)}$, and (2.11) obtains the form

$$f^{[q+1]}(x_1\bullet x_2\bullet\dots\bullet x_{q+1})=f(f(x_1)f(x_2)\dots f(x_{q+1})).$$

3. THE GENERAL ASSOCIATIVE LAW

Below we assume that (Q,f) is an (n,m)-semigroup. We will prove the following proposition in several steps.

Theorem 3.1. *For every positive integer r, $(Q,f^{(r)})$ is an $(m+rk,m)$-semigroup.*

(For $r=1$, the theorem is true by assumption.)

First, we will prove the following:

Lemma 3.2. *Let $r\geq 2$, $x_1\in Q^{(n)}$ and $x_\nu\in Q^{(k)}$ for each $\nu:2\leq\nu\leq r$, and let $x_i=y_ia$, $x_j=by_j$, $a,b\in Q$, for given i,j, such that $1\leq i<j\leq r$.* Then

$$f^r(x_1,x_2,\dots,x_r)=f^r(x_1',x_2',\dots,x_r'), \tag{3.1}$$

where $x_\nu'=x_\nu$ for $\nu\neq i,j$, and $x_i'=y_ib$, $x_j'=ay_j$.

Proof. It suffices only to prove the case $j=i+1$.

For $r=2$, (3.1) is in fact (2.8).

Suppose that (3.1) is true for a given $r\geq 2$. To show that (3.1) is also true when r is changed to $r+1$, we will use the equality (2.3) for $p+q=r+1$. If $i<r$, i.e. $i+1<r+1$, then we put $q=i+1$ in (2.3) and use the inductive hypothesis. For $i=r$, we put $q=1$ in (2.3) and we again use the inductive hypothesis. □

Using this lemma in almost the same way as the proof of the part $(ii)\Rightarrow(i)$ in Proposition 2.4, one can show the following:

Proposition 3.3. *For each positive integer r, $(Q,f^{(r)})$ is an $(m+rk,m)$-groupoid.* □

It remains to be shown that, for each $r\geq 2$, $(Q,f^{(r)})$ is an $(m+rk,m)$-semigroup, i.e., that

$$f^{(r)}(f^{(r)}(xa)by)=f^{(r)}(f^{(r)}(xb)ay) \tag{3.2}$$

holds for every $a,b\in Q$, $x,y\in Q^{(+)}$, such that $|xa|=m+rk$, $|by|=rk$.

Therefore, representing xa and by in the form

$$xa = x_1 x_2 \cdots x_r a,\ \ by = by_1 y_2 \cdots y_r,$$

where $|x_1| = n, |x_2| = \cdots = |x_{r-1}| = |x_r a| = k = |by_1| = |y_2| = \cdots = |y_r|$, we use (2.4) and (2.3) to represent the left-hand side of (3.2) in the form

$$f^{2r}(x_1, x_2, \ldots, x_{r-1}, x_r a, by_1, y_2, \ldots, y_r)\,. \qquad (3.3)$$

Then, applying equality (3.1) in (3.3), and performing a reverse procedure from the previous one, we would get the right-hand side of (3.2).

This completes the proof of Theorem 3.1.

The equality in the following proposition is known as the *General Associative Law* (GAL) for fully commutative (n, m)-semigroups.

Proposition 3.4. *Let* (Q, f) *be an* (n, m)*-semigroup. Then*

$$f^{(r)}(f^{(p_1)}(x_1)\, f^{(p_2)}(x_2) \cdots f^{(p_s)}(x_s)\; y) = f^{(r+p_1+p_2+\cdots+p_s)}(x_1 x_2 \cdots x_s\; y), \qquad (3.4)$$

where $|x_\nu| = m + kp_\nu$ *for* $1 \le \nu \le s$, $sm + |y| = m + rk$, *and* r, p, s *are given positive integers. (Here, it is possible that* $|y| = 0$, *i.e.* $y = e$*).*

Proof. For $s = 1$, (3.4) has the form

$$f^{(r)}(f^{(p)}(x)y) = f^{(r+p)}(xy) \qquad (3.5)$$

(where p stands for p_1, and x for x_1) which can be proved using (2.4) and (2.3). Then, supposing that (3.4) is true for some $s-1 \ge l$, we substitute $f^{(p_1)}(x_1)y$ with z in the left-hand side of (3.4), apply first the inductive hypothesis and then (3.5), we obtain (3.4). □

Remark 1. Fully commutative $(m + k, m)$-groups are defined in [2]. Namely, a fully commutative $(m + k,$ m$)$-semigroup is a *fully commutative* $(m + k, m)$*-group* if for each $a \in Q^{(k)}$, $b \in Q^{(m)}$ there is an $x \in Q^{(m)}$ such that $f(ax) = b$. These are also considered in: [7], [8], [9], [6]. In [2] it is shown that every infinite set is a carrier of a fully commutative $(m + k, m)$-group, that for $m \ge 2$ there is no finite fully commutative $(m + k, m)$-group with more than two elements, and that there are non-isomorphic fully commutative $(m + k, m)$-groups with a two-element carrier. In [7] it is noted that a structure of a fully commutative $(m + k, m)$-group can be built on every algebraically closed field, and in [8] the class of affine and the class of projective $(m + k, m)$-groups are defined. In [9], the class of affine and the class of projective $(m + k, m)$-groups with sets of complex numbers as their carriers are studied. We also note that in [5], [8] and [10] several axiom systems of the class of $(m + k, m)$-groups are obtained.

Following the idea of [10], a proposition analogous to Theorem 1 in [5] for fully commutative $(m + k, m)$-groups can be stated and proved.

Remark 2. The class of fully commutative $(m + k, m)$-groups is a subclass of the class of fully commutative $(m + k, m)$-quasigroups, defined in [3]. Namely, a fully commutative $(m + k, m)$-groupoid (Q, f) is said to be a $(m + k, m)$-quasigroup iff for each $x \in Q^{(k)}$, $y \in Q^{(m)}$ there is a unique $z \in Q^{(m)}$, such that $f(xz) = y$. It is shown in [3] that each cancellative fully commutative $(m + k, m)$-groupoid is embeddable in a fully commutative $(m + k, m)$-quasigroup. It is also shown there that, if $q \ge 3$, then there is a fully commutative $(q, q-1)$-quasigroup with $q + 1$ elements. On the other hand, there is no fully commutative $(m + k, m)$-quasigroup with $q + 1$ elements for $2 \le q \le m$. We note that there are many open problems on finite fully commutative $(m + k, m)$-quasigroups.

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Резиме

ЗА ПОТПОЛНО КОМУТАТИВНИТЕ (*n*, *m*)-ГРУПОИДИ

Во неколку трудови (на пример [2] и [8]) за потполно комутативни векторско вредносни групоиди се разгледуваат соодветни својства со помош на комутативни векторско вредносни групоиди. Главната цел на оваа работа е „автономно" испитување на класата потполно комутативни векторско вредносни групоиди, т.е. испитување без да се напушти самата класа.