VARIETIES OF GROUPOIDS WITH AXIOMS OF THE FORM $x^{m+1}y = xy$ AND/OR $xy^{n+1} = xy$

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Abstract. The subject of this paper are varieties $\mathcal{U}(M;N)$ of groupoids defined by the following system of identities

$$\{x^{m+1}\cdot y = xy : m \in M\} \cup \{x \cdot y^{n+1} = xy : n \in N\},$$

where M, N are sets of positive integers. The equation $\mathcal{U}(M;N) = \mathcal{U}(M';N')$ for any given pair (M,N) is solved, and, among all solutions, one called canonical, is singled out. Applying a result of Evans ([6]) it is shown for finite M and N that: if M and N are nonempty and $\gcd(M) = \gcd(M \cup N)$, or only one of M and N is nonempty, then the word problem is solvable in $\mathcal{U}(M;N)$.

0 Introduction

A *groupoid* is an algebra $\mathbf{G} = (G, \bullet)$ with one binary operation $\bullet : (a,b) \mapsto ab$. (We will often omit the operation sign.) Assuming the usual meanings of other algebraic notions, we do not define them explicitly.

By a result of P. Hall (see, for example, [3], III.2, Ex.2, p.125,or [10], pp 39-40), for any positive integer k there exist $\frac{(2k-2)!}{k!(k-1)!}$ k-th *groupoid powers* $x \mapsto x^k$. In this paper, we assume the groupoid power x^k defined as follows:

$$x^1 = x, \ \ x^{k+1} = x^k x.$$

So $x^3 = x^2x = (xx)x$.

A formula $x^{k+1}y = xy$ ($xy^{k+1} = xy$), will be called a *left* (*right*) *equation*. (Here, and further on, m, n, k, p, i, j, s are assumed to be positive integers, and xy^{n+1} stands for $x \cdot y^{n+1}$, and $x^{m+1}y$ for $x^{m+1} \cdot y$.) The varieties $\mathcal{U}(M;\emptyset)$, $\mathcal{U}(\emptyset;N)$, $\mathcal{U}(M;N)$, where $M \neq \emptyset$ and $N \neq \emptyset$, are said to be *left, right, two-sided*, respectively. (Throughout the paper "variety" will mean "left, right, or two–sided variety".)

Below, $\mathcal{U}(m_1, m_2, m_3, \cdots; n_1, n_2, n_3, \cdots)$ will be an abbreviation for $\mathcal{U}(\{m_1, m_2, m_3 \cdots\}; \{n_1, n_2, n_3, \cdots\})$.

The paper consists of three sections. In Section 1 we show that each variety $\mathcal{U}(M;N)$ admits a canonical axiom system. In Section 2 we solve the equation $\mathcal{U}(M;N) = \mathcal{U}(M';N')$. Finally, in Section 3, we consider "incomplete $\mathcal{U}(M;N)$-groupoids", and applying a result of Evans ([6]) we show that the word problem is solvable in $\mathcal{U}(M;N)$ for finite M and N in each of the cases: (i) $N = \emptyset$,(ii) $M = \emptyset$, (iii) $M \neq \emptyset$, $N \neq \emptyset$, $\gcd(M)=\gcd(M \cup N)$.[1]

1 A Canonical Axiom System for $\mathcal{U}(M;N)$

The main result of this section is the following

Theorem 1 *If M, N are nonempty sets of positive integers, then*

(*l*) $\mathcal{U}(M;\emptyset) = \mathcal{U}(\gcd(M);\emptyset)$.

(*r*) $\mathcal{U}(\emptyset;N) = \mathcal{U}(\emptyset;\langle N\rangle)$.[2]

(*t*) $\mathcal{U}(M;N) = \mathcal{U}(\gcd(M);\gcd(M \cup N))$.

In order to prove this theorem we will show some lemmas, where m,n,k,p,i,j,s are assumed to be positive integers as above, and q a nonnegative integer.

Lemma 1.1 *If $1 \le k \le m$, then*

$$\mathcal{U}(m;\emptyset) \models x^{qm+k+1} = x^{k+1}.\ ^{3}$$

Proof. Clearly, $x^{m+2} = x^2, \cdots, x^{2m+1} = x^{m+1}$ are true in $\mathcal{U}(m;\emptyset)$; then the proof follows by induction on q and k. □

As a corollary, we obtain:

[1] $\gcd(M)$ is the greatest common divisor of M

[2] $\langle N\rangle$ is the additive groupoid of integers generated by N.

[3] $\mathcal{V} \models \tau_1 = \tau_2$ means: the equation $\tau_1 = \tau_2$ is true in the variety $\mathcal{V}$.

Lemma 1.2 *If $m|n$, then $\mathcal{U}(m;\emptyset) \subseteq \mathcal{U}(n;\emptyset)$.* [4] □

Lemma 1.3 *If $\gcd(M) = d \notin M$, then there exists a nonempty set M_1 of positive integers such that*

$$\mathcal{U}(M;\emptyset) = \mathcal{U}(M_1;\emptyset),\ d = \gcd(M_1),\ \min(M_1) < \min(M).\ ^{5} \qquad (1)$$

Proof. Let $p = \min(M)$. The assumption $d \notin M$ implies that $d < p$ and thus there exists an $n \in M$ such that p is not a divisor of n. Then $n = qp + k$, $d|k$, $k < p$ and, if $M_1 = (M \setminus \{n\}) \cup \{k\}$, the relations (1.1) hold. □

As a corollary of Lemma 1.2 and Lemma 1.3 we obtain the equality (l).

The equality (r) is an obvious corollary of the following

Lemma 1.4 $\mathcal{U}(\emptyset; m, n) \subseteq \mathcal{U}(\emptyset; m+n)$.

Proof. $\mathcal{U}(\emptyset; m) \models (x^{m+1})^i = x^{m+i}$, and therefore $\mathcal{U}(\emptyset; m, n) \models (x^{m+1})^{n+1} = x^{m+n+1}$. Thus, if $\mathbf{G} \in \mathcal{U}(\emptyset; m, n)$, then:

$$x^{m+n+1}y = (x^{m+1})^{n+1} = x^{m+1}y = xy, \text{ i.e. } \mathbf{G} \in \mathcal{U}(\emptyset; m+n). \ \square$$

It remains to prove (t).

Lemma 1.5 *If $L = \{\gcd(m, n) : n \in N\}$, then $\mathcal{U}(m; N) = \mathcal{U}(m; L)$.*

Proof. By a similar argument as in Lemma 1.2, $\mathcal{U}(m; L) \subseteq \mathcal{U}(m; N)$. If $n \in N$ and $d = \gcd(m, n)$, then there exist i, j such that $im + d = jn$. By Lemma 1.1, $\mathcal{U}(m; n) \models x^{d+1} = x^{im+d+1}$, and therefore $\mathcal{U}(m; n) \models xy^{d+1} = xy$. □

In completing the proof of (t) we will use the following result (for example [5] or [9]).

Lemma 1.6 *If S is an additive groupoid of positive integers and $d = \gcd(S)$, then:*

(i) $\gcd(N) = d$ *for any generating subset N of S.*

(ii) *There exists the least generating subset $K = \{n_1, n_2, \cdots, n_k\}$ of S, and K is finite.*

(iii) *There exists $s \in S$ such that for each positive integer j, $s + jd \in S$.* □

Lemma 1.7 *If $d_1, d_2, \cdots, d_k$ are divisors of m, and $d = \gcd(d_1, d_2, \cdots, d_k)$, then*

$$\mathcal{U}(m; d_1, d_2, \cdots, d_k) = \mathcal{U}(m; d).$$

Proof. The inclusion $\mathcal{U}(m; d) \subseteq \mathcal{U}(m; d_1, d_2, \cdots, d_k)$ follows as in Lemma 1.5. For the converse inclusion, denote by S the additive groupoid of positive integers generated by $\{d_1, d_2, \cdots, d_k\}$. By Lemma 1.6 (i) and (r) we have $\gcd(S) = d$, and $\mathcal{U}(m; d_1, d_2, \cdots, d_k) = \mathcal{U}(m; S)$. By Lemma 1.6 (iii) there exists $s \in S$ such that $ms + d \in S$ and thus, by Lemma 1.1, $\mathcal{U}(m; S) \models y^{ms+d+1} = y^{d+1}$. □

Finally, by (l), (r), Lemma 1.5 and Lemma 1.7, it follows that

$$\mathcal{U}(M; N) = \mathcal{U}(m; n),$$

where $m = \gcd(M)$ and $n = \gcd(M \cup N)$. This completes the proof of (t).

We note that the following equality holds in $\mathcal{U}(m; m)$

$$(x^{m+1})^{m+1} = x^{m+1}, \tag{2}$$

(or more generally, in $\mathcal{U}(m; n)$, where $n|m$, the equality $(x^{in+1})^{m+1} = x^{in+1}$ holds.)

The results obtained in Theorem 1 suggest saying that

$$x^{m+1}y = xy, \ \{xy^{n+1} = xy : n \in K\}, \ \{x^{m+1}y = xy, xy^{n+1} = xy\}$$

is the *canonical axiom system* of $\mathcal{U}(M; \emptyset)$, $\mathcal{U}(\emptyset; N)$, $\mathcal{U}(M; N)$, respectively, where M, N are nonempty sets of positive integers, $m = \gcd(M)$, K is the least generating subset of $\langle N \rangle$, and $n = \gcd(M \cup N)$.

As a corollary of Theorem 1 (for example [2]) we obtain

Corollary 1.1 *For any pair (M, N) the variety $\mathcal{U}(M; N)$ is finitely based.* □

2 Closed Sets of Equations in $\mathcal{U}(M;N)$

The main result of this section is the following

Theorem 2 *If M, N, M', N' are nonempty sets of positive integers, then:*

(i) $\mathcal{U}(M; \emptyset) = \mathcal{U}(M'; \emptyset) \iff \gcd(M) = \gcd(M')$.

(ii) $\mathcal{U}(\emptyset; N) = \mathcal{U}(\emptyset; N') \iff \langle N \rangle = \langle N' \rangle$.

(iii) $\mathcal{U}(M; N) = \mathcal{U}(M'; N') \iff \gcd(M) = \gcd(M') \,\&\, \gcd(M \cup N) = \gcd(M' \cup N')$.

[4] $m|n$ denotes that m is a divisor of n.

[5] $\min(M)$ denotes the least element in M.

(iv) $\mathcal{U}(M; \emptyset) \neq \mathcal{U}(\emptyset; N)$; $\mathcal{U}(M; \emptyset) \neq \mathcal{U}(M'; N')$; $\mathcal{U}(\emptyset; N) \neq \mathcal{U}(M'; N')$.

The $\Leftarrow$-parts of (i), (ii), (iii) hold by Theorem 1. The corresponding $\Rightarrow$-parts and (iv) are corollaries of the following statement, shown in [4] (Proposition 3.5).

Proposition 2.1 *Let $\mathbf{H}$ be a free groupoid in the variety $\mathcal{U}(M; N)$. Then the following statements hold:*

(i) *If $M \neq \emptyset$, $N = \emptyset$, $\gcd(M) = m$, then a left equation $x^{n+1}y = xy$ holds in $\mathbf{H}$ iff $m|n$; no right equation holds in $\mathbf{H}$.*

(ii) *If* $M = \emptyset$, $N \neq \emptyset$, *then a right equation* $xy^{n+1} = xy$ *holds in* $\mathbf{H}$ *iff* $n \in \langle N \rangle$*; no left equation holds in* $\mathbf{H}$.

(iii) *If* $M \neq \emptyset$, $N \neq \emptyset$ *and* $m = \gcd(M)$, $n = \gcd(M \cup N)$, *then* $x^{i+1}y = xy$ *iff* $m|i$, *and* $xy^{j+1} = xy$ *iff* $n|j$, *hold in* $\mathbf{H}$. □

(We note that only-if parts of (i) and (iii) in Proposition 2.1 follow from the fact that $C_n \in \mathcal{U}(n; \emptyset) \cap \mathcal{U}(kn; n)$, where C_n is the groupoid that is the reduction of the cyclic group of order n to its binary operation.)

A set Σ of equations is said to be *closed* if, for every equation ε, the following implication holds:

$$(\Sigma \models \varepsilon) \Rightarrow (\varepsilon \in \Sigma).$$

Proposition 2.2 (i) *Assume that* Σ *is a set of equations containing at least one left equation and at least one right equation. Then* Σ *is a closed set iff there exist two positive integers* m *and* n *such that* n *is a divisor of* m *and*

$$\Sigma = \{x^{im+1}y = xy : i \geq 1\} \cup \{xy^{jn+1} = xy : j \geq 1\}.$$

(ii) *A set* Σ *of left equations is closed iff there is a positive integer* m *such that*

$$\Sigma = \{x^{im+1}y = xy : i \geq 1\}.$$

(iii) *A set* Σ *of right equations is closed iff there is an additive groupoid* S *of positive integers such that*

$$\Sigma = \{xy^{n+1} = xy : n \in S\}. \ \square$$

The lattices $\mathcal{U}_l, \mathcal{U}_r, \mathcal{U}$ (of all left, right, two–sided varieties, respectively) can be characterized as follows:

Proposition 2.3 (l) $\mathcal{U}_l$ *is isomorphic to the lattice of positive integers, where* $m \leq n$ *iff* $m|n$.

(r) $\mathcal{U}_r$ *is antiisomorphic to the lattice of additive groupoids of positive integers.*

(t) $\mathcal{U}$ *is isomorphic to the lattice of pairs* (m, n) *ofpositive integers such that* n *is divisor of* m*, and:*

$$(m, n) \leq (m', n') \iff m|m' \ \&\ n|n'. \ \square$$

3 Incomplete $\mathcal{U}(M; N)$– Groupoids and Varieties $\mathcal{U}(M; N)$ with Solvable Word Problem

We investigate here the class of incomplete $\mathcal{U}(M; N)$–groupoids and by applying the main result of Evans's paper [6], we solve the word problem for some varieties $\mathcal{U}(M; N)$.

The term "*incomplete groupoid*" ([6]) has the same meaning as "*halfgroupoid*" ([1]) or "*partial groupoid*" ([8]). Namely, if G is a nonempty set, D a subset of $G \times G$, and $\cdot : (x, y) \mapsto xy$ a map from D into G, then the pair $\mathbf{G} = (G, \cdot)$ is called an *incomplete groupoid* with the *domain* D.

A groupoid $\mathbf{H} = (H, \bullet)$ is called an *extension* of the incomplete groupoid $\mathbf{G}$ iff $G \subseteq H$ and $a \bullet b = ab$, for every $(a, b) \in D$. If $G^o = G \cup \{0\}$, where $0 \notin G$, then the groupoid $\mathbf{G}^o = (G^o, \bullet)$ defined by

$$x \bullet y = \begin{cases} xy, & \text{if } (x, y) \in D \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

is an extension of $\mathbf{G}$. We call $\mathbf{G}^o$ the *trivial extension* of $\mathbf{G}$.

If M, N are sets of positive integers such that $M \cup N \neq \emptyset$, then we denote by $\mathcal{IU}(M; N)$ the *class of incomplete groupoids* $\mathbf{G}$, such that the corresponding trivial closure $\mathbf{G}^o$ satisfies the following implications:

$$x^{m+1} \in G \Rightarrow x^{m+1} \bullet y = x \bullet y, \quad y^{n+1} \in G \Rightarrow x \bullet y^{n+1} = x \bullet y, \tag{4}$$

for any $m \in M, n \in N, x, y \in G$.

Let $\mathbf{G}$ be an incomplete groupoid and K a set of positive integers. We define an *equivalence* $\sim_K$ on G as follows. If $K = \emptyset$, then $\sim_K$ is the equality on G. If $K \neq \emptyset$, we define a relation $\rightarrow_K$ on G by:

$$c \rightarrow_K d \iff d = c^{k+1}, \tag{5}$$

for $c, d \in G$ and some $k \in K$, and we put: $c \leftrightarrow_K d \iff (c \rightarrow_K d \text{ or } c \leftarrow_K d)$. We denote by $\sim_K$ the reflexive, symmetric and transitive closure of $\rightarrow_K$ on G, i.e., the equivalence on G generated by $\rightarrow_K$.

By (3), (4), and (5), we obtain the following characterization of the class $\mathcal{IU}(M;N)$:

$$\mathbf{G} \in \mathcal{IU}(M; N) \Leftrightarrow (\forall x, x', y, y' \in G)(x \sim_M x' \ \&\ y \sim_N y' \Rightarrow xy = x'y') \tag{6}$$

Let $\mathbf{G} \in \mathcal{IU}(M; N)$ and define

$$A = \{a \in G \mid a^{k+1} \in G, \text{ for every } k \in M \cup N\}, \ B = G \setminus A; \tag{7}$$

clearly, $B = \{b \in G \mid b^{k+1} \notin G, \text{ for some } k \in M \cup N\}$.

By (3), (4) and (7) it follows that

$$\mathbf{G} \in \mathcal{IU}(M;N) \ \& \ A = G \Rightarrow \mathbf{G}^o \in \mathcal{U}(M;N). \tag{8}$$

Note that, in the special case when $M = \{m\}, N = \{n\}$, and $n|m$, we have $A = \{a \in G \mid a^{m+1} \in G\}$ and $B = \{b \in G \mid b^{m+1} \notin G\}$.

The following proposition is true.

Proposition 3.1 (i) *If $\mathbf{G} \in \mathcal{IU}(m;\emptyset)$, then for each $a \in A, q \geq 0$, and $1 \leq k \leq m$, the equality $a^{qm+k+1} = a^{k+1}$ holds.*

(ii) *If $\mathbf{G} \in \mathcal{IU}(m;n), n|m$, and $a \in A$, then $(a^{in+1})^{m+1} = a^{in+1}$.*

(iii) *$\mathcal{IU}(\emptyset; r, i) = \mathcal{IU}(\emptyset; r, i, r+i)$.* □

Using (5) and Proposition 3.1 we obtain the following

Lemma 3.1 *Let $\mathbf{G} \in \mathcal{IU}(m;n)$ and $n|m$. Then*

(i) *$x \sim_m y \ \Rightarrow \ x^{m+1} = y^{m+1}$;*

(ii) *$x \sim_m y \ \Rightarrow \ x, y \in A \vee x = y \in B$, where $\sim_m$ stands for $\sim_{\{m\}}$.*

Proof. Let $x \sim_m y$. If $x = y$, then $x^{m+1} = y^{m+1}$. If $x \neq y$, then $x \sim_m y \iff (\exists t_0, t_1, \cdots t_s \in G) x = t_0 \leftrightarrow t_1 \leftrightarrow \cdots \leftrightarrow t_s = y$, where $\leftrightarrow$ stands for $\leftrightarrow_{\{m\}}$. The proof is given by induction on s. If $s = 1$, then $x^{m+1} = y^{m+1}$, and $x, y \in A$. If $s = 2$, we have the following four cases:

1) $x \to t \to y$; then $t = x^{m+1}, \ y = t^{m+1}, \ y = (x^{m+1})^{m+1} = x^{m+1}$ (by Proposition 3.1), and thus $y^{m+1} = (x^{m+1})^{m+1} = x^{m+1}$;
2) $x \to t \leftarrow y$; then $x^{m+1} = t = y^{m+1}$;
3) $x \leftarrow t \leftarrow y$; then $x^{m+1} = y^{m+1}$ follows by symmetry of 1);
4) $x \leftarrow t \to y$; then $x = t^{m+1} = y$;

and in each case $x, y \in A$.

If $s > 2$, then applying 1)–4), the sequence $t_0, t_1, \cdots, t_s$ can be reduced to a sequence with less than $s+1$ elements. □

As a corollary of Lemma 3.1 we obtain the following

Proposition 3.2

$$\mathbf{G} \in \mathcal{IU}(m;n) \ \& \ n|m \Rightarrow (\forall b, b' \in B)(b \sim_m b' \Rightarrow b = b'). \ \square \tag{9}$$

If $b \in B$, then we denote by $p(b)$ the positive integer p, such that

$$b^p \neq 0, \ b^{p+1} = 0. \tag{10}$$

Now we are ready to prove the main result.

Theorem 3 *If the pair (M, N) satisfies one of the following conditions (i) $M = \emptyset, \ N \neq \emptyset$; (ii) $M = \{m\}, \ N = \emptyset$; (iii) $M = \{m\} = N$, then for each (finite) $\mathbf{G} \in \mathcal{IU}(M;N)$ there exists a (finite) $\mathbf{H} \in \mathcal{U}(M;N)$ that is an extension of $\mathbf{G}$.*

Proof. If $B = \emptyset$, then $\mathbf{G}^o$ is an extension of $\mathbf{G}$, finite if G is finite, such that, by (8), $\mathbf{G}^o \in \mathcal{U}(M;N)$. Thus, it remains to build an extension $\mathbf{H} = (H, \bullet) \in \mathcal{U}(M;N)$, assuming that $B \neq \emptyset$.

Consider first the case (i): $M = \emptyset, \ N \neq \emptyset$.

Let L be a set such that $L \cap G^o = \emptyset$, and let $b \mapsto \underline{b}$ be a surjection from B onto L with the following property:

$$(\forall b, c \in B)(\underline{b} = \underline{c} \iff b \sim c \ \& \ b^p = c^q), \tag{11}$$

where $\sim$ is an abbreviation for $\sim_N$, $p = p(b), \ q = p(c)$. Define an operation $\bullet$ on $H = G^o \cup L$ as follows:

1) If $x, y \in G, \ b \in B$, then:
   1.1) $x \bullet y = xy$, for $xy \in G$;
   1.2) $x \bullet y = \underline{b}$, for $x = b^p, y \sim b$.
2) If $x \in G, b \in B$, then:
   2.1) $\underline{b} \bullet x = \underline{b}$, for $x \sim b$;
   2.2) $x \bullet \underline{b} = x \bullet b$, if $x \bullet b$ is defined by 1.1) or 1.2).
3) If $b, c \in B$, and $b \sim c$, then $\underline{b} \bullet \underline{c} = \underline{b}$.
4) $x \bullet y = 0$, in any other case.

Using (11) and (6) one can directly show that $\bullet$ is a well-defined operation on H.

It follows by 1.1) that $\mathbf{H}$ is an extension of $\mathbf{G}$, and so it remains to show that $\mathbf{H} \in \mathcal{U}(\emptyset; N)$.

First, by (11) and the definition of $\bullet$ we obtain the following properties:

5) If $a \in A, \ b \in B, \ z \in L \cup \{0\}, \ n \in N, \ p = p(b)$, then:
   5.1) $a_\bullet^{n+1} = a^{n+1}$;
   5.2) $b_\bullet^{n+1} = b^{n+1}$, for $n + 1 \leq p$;
   5.3) $b_\bullet^{n+1} = \underline{b}$, for $n + 1 > p$;
   5.4) $z_\bullet^k = z$, for each $k \in Z^+$.

(Here, $y_\bullet^k$ is the k–th power of y in $\mathbf{H}$, i.e. $y_\bullet^1 = y,\ y_\bullet^{k+1} = y_\bullet^k \bullet y$.)

Now, by using properties 5) and the definition of $\bullet$, we can show that:

6) $x \bullet (y_\bullet^{n+1}) = x \bullet y$, for each $x, y \in H,\ n \in N$, i.e. $\mathbf{H} \in \mathcal{U}(\emptyset; N)$.

Thus we have proved Theorem 3 in the case (i).

Now, consider the cases (ii) $M = \{m\},\ N = \emptyset$ and (iii) $M = N = \{m\}$. The construction of a groupoid $\mathbf{H} \in \mathcal{U}(M; N)$ that is an extension of $\mathbf{G} \in \mathcal{IU}(M; N)$ is formally the same in case (ii) as in case (iii). In both cases we will denote the equivalence $\sim_M$ in G by $\sim$; and $\approx$ is the equality in G in case (ii), and $\approx$ is the same as $\sim$ in case (iii).

Let
$$L = \{(b, i) : b \in B, p(b) < i \leq m\}$$
and $H = G^o \cup L$. (The union defining H is assumed to be disjoint.)

Define an operation $\bullet$ in H as follows.

1′) If $x, y \in G$, then:

1.1′) $x \bullet y = xy$, if $xy \in G$;

1.2′) $x \bullet y = b$, if $b \in B,\ x \sim b^m,\ p(b) = m,\ y \approx b$;

1.3′) $x \bullet y = (b, p(b) + 1)$, if $x \sim b^{p(b)},\ p(b) < m,\ y \approx b$.

2′) If $b \in B,\ y \in G,\ y \approx b$, then:

2.1′) $(b, m) \bullet y = b$;

2.2′) $(b, i) \bullet y = (b, i + 1)$, if $p(b) < i < m$.

3′) If $x \in L$, then $x \bullet x = x$.

4′) $x \bullet y = 0$, in any other case.

Thus we obtain an extension $\mathbf{H} = (H, \bullet)$ of $\mathbf{G}$. (The product $x \bullet y$ for *(ii)* in the cases 1.2′) and 1.3′) is well–defined by (9).)

It remains to show that $\mathbf{H} \in \mathcal{U}(M; N)$.

For that purpose, note first that the following statements hold.

5′) If $a \in A,\ x \in B \cup L \cup \{0\}$, then

5.1′) $a_\bullet^{m+1} = a^{m+1} \in G$;

5.2′) $x_\bullet^{m+1} = x$.

(Here, as in 5), $y_\bullet^k$ is the k–th power of y in $\mathbf{H}$.)

We will now show that:

6′) $x_\bullet^{m+1} \bullet y = x \bullet y$, for any $x, y \in H$.

Namely, if $x \in B \cup L \cup \{0\}$ or $y \in L \cup \{0\}$, then the equality 6′) follows from 3′), 4′) and 5.2′). There remains the case $x \in A,\ y \in G$. Here, by 5.1′) and the definition 1.1′), 1.2′), 1.3′) and 4′), we obtain the desired equality 6′).

Hence (in the case $M = \{m\},\ N = \emptyset$), $\mathbf{H} \in \mathcal{U}(m; \emptyset)$.

It remains to show that, for $M{=}N{=}\{m\}$, the following identity holds in $\mathbf{H}$:

7′) $x \bullet (y_\bullet^{m+1}) = x \bullet y$.

By the same reasoning as for 6′), the equality 7′) is true whenever $y \in B \cup L \cup \{0\}$ or $x = 0$. For $x \in G \cup L$ and $y \in A$, one can show that 7′) is also true, in the same way as for 6′).

Hence (in the case $M = N = \{m\}$), $\mathbf{H} \in \mathcal{U}(m; m)$, and this completes the proof of Theorem 3.

The following statement is a special case of the main result of the paper [6]:

Proposition 3.3 *If the pair (M, N) is such that for every $\mathbf{G} \in \mathcal{IU}(M; N)$ there exists an extension $\mathbf{H} \in \mathcal{U}(M; N)$, then the word problem is solvable in the variety $\mathcal{U}(M; N)$.* □

As a corollary of Theorem 1, Proposition 3.3 and Theorem 3, we obtain the following

Theorem 4 *If $M \cup N$ is finite and one of the following conditions holds:*

(i) $N = \emptyset$; (ii) $M \neq \emptyset, N \neq \emptyset$, *and* $\gcd(M) = \gcd(M \cup N)$; (iii) $M = \emptyset$, *then the word problem is solvable in the variety $\mathcal{U}(M; N)$.* □

Remarks

1. Theorem 1 and Theorem 2 suggest the following two questions:

a) Is the implication
$$\mathcal{U}(M; N) = \mathcal{U}(M'; N') \Rightarrow \mathcal{IU}(M; N) = \mathcal{IU}(M'; N')$$
true?

b) Is it true that, for every pair (M, N), every $\mathbf{G} \in \mathcal{IU}(M; N)$ has an extension $\mathbf{H} \in \mathcal{U}(M; N)$?

The answer to both questions, in general, is negative, as the following example shows.

Let M be a nonempty set of positive integers, $\gcd(M) = m$ and $G = \{1, 2, \cdots, m + 1, m + 2\}$. Let $\mathbf{G} = (G, \bullet)$ be an incomplete groupoid such that the corresponding canonical extension $\mathbf{G}^o$ is defined as follows:

$a_1)$ $i \bullet 1 = i+1$, if $i = 1, 2, \cdots, m+1$;
$a_2)$ $1 \bullet (m+2) = 1$;
$a_3)$ $(m+1) \bullet (m+2) = m+1$;
$a_4)$ $x \bullet y = 0$, otherwise.

If $m \notin M$ and $p = \min(M) > m+1$, then $x^{n+1} = 0$ for every $x \in G$, $n \in M$, and thus, by (3.3), $\mathbf{G} \in \mathcal{IU}(M;\emptyset)$. On the other hand, we have $1_\bullet^{m+1} \bullet 1 = (m+1) \bullet 1 = m+2 \neq 2 = 1 \bullet 1$, which implies that $\mathbf{G} \notin \mathcal{IU}(m;\emptyset)$. Hence, $\mathcal{IU}(m;\emptyset) \not\subseteq \mathcal{IU}(M;\emptyset)$, i.e. the answer to the question $a)$ is negative.

Also, $\mathbf{G} \in \mathcal{IU}(M;\emptyset)$ cannot be embedded in an $\mathbf{H} \in \mathcal{U}(M;\emptyset)$ $(= \mathcal{U}(m;\emptyset))$, because $(1_\bullet^{m+1}) \bullet 1 = m+2 \neq 2 = 1 \bullet 1$.

2. Theorem 3 and the main result of [7] imply that, for each of the cases: i) $M \neq \emptyset$, $N = \emptyset$; ii) $M \neq \emptyset \neq N$, $\gcd(M) = \gcd(M \cup N)$; iii) $M = \emptyset$, $N \neq \emptyset$, the embeddability problem: "For a finite $\mathbf{G} \in \mathcal{IU}(M;N)$, is there an extension $\mathbf{H} \in \mathcal{U}(M;N)$?" is solvable.

3. In connection with Theorem 4, the authors conjecture that, applying the main result of [7], one can obtain the following variant of **Theorem 4**: "*If $M \cup N$ is finite, then the word problem is solvable in $\mathcal{U}(M;N)$.*"

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