

ON MONOASSOCIATIVE GROUPOIDS

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Abstract

The subject of this paper is the variety (denoted by $Mass$) of mono-associative groupoids, i.e. groupoids in which every cyclic subgroupoid is a subsemigroup. A description of free objects in $Mass$ is given. Using a convenient definition of injective groupoids in $Mass$, it is shown that a groupoid \underline{H} is free in $Mass$ iff \underline{H} is injective in $Mass$ and the set of prime elements in \underline{H} generates \underline{H} . (This property is named Bruck Theorem for $Mass$.) Neither of the classes $Mass_{inj}$ (injective objects in $Mass$) and $Mass_{fr}$ (free objects in $Mass$) is hereditary. A characterization of free subgroupoids of a groupoid $\underline{H} \in Mass_{fr}$ is obtained. It is shown that every groupoid $\underline{H} \in Mass_{fr}$ with a two-element basis has a subgroupoid $\underline{Q} \in Mass_{fr}$ with an infinite basis.

1. Preliminaries

A groupoid is a pair $\underline{G} = (G, \cdot)$, where G is a nonempty set and " \cdot " is a mapping $(x, y) \mapsto xy$, from G^2 into G . \underline{G} is said to be *injective* iff:

$$(\forall x, y, u, v \in G)(xy = uv \Rightarrow (x, y) = (u, v)). \quad (1.1)$$

An element $a \in G$ is *prime*¹ in \underline{G} iff $a \notin GG$, where

$$GG = \{xy \mid x, y \in G\}. \quad (1.2)$$

The following statement is well known (for example; [1; L.1.5]).

Proposition 1.1 (Bruck Theorem). A groupoid $\underline{F} = (F, \cdot)$ is absolutely free (i.e. free in the variety of groupoids) iff the following conditions hold:

a) \underline{F} is injective.

b) The set B of primes in \underline{F} is nonempty and generates \underline{F} .

Below we assume that \underline{F} is a given absolutely free groupoid with the basis B . The *length* $|v|$, the *set* $P(v)$ of parts and the *content* $cn(v)$ of an element $v \in F$, are defined as follows:

$$\begin{aligned} |b| &= 1, & |tu| &= |t| + |u|; & P(b) &= \{b\}, & P(tu) &= \{tu\} \cup P(t) \cup P(u); \\ cn(b) &= \{b\}, & cn(tu) &= cn(t) \cup cn(u), \end{aligned} \quad (1.3)$$

for any $b \in B, t, u \in F$.

We will also use an absolutely free groupoid $\underline{E} = (E, \cdot)$ with a one-element basis $\{e\}$, assuming that $F \cap E = \emptyset$. Elements of E will be denoted by f, g, h, \dots and will be called (*groupoid*) *powers*. It should be noted that (1.3) makes meaningful notions "the length $|f|$ " and "the set $P(f)$ of parts" of an element $f \in E$.

If \underline{G} is a groupoid, then each $f \in E$ induces a transformation $f^{\underline{G}} : G \rightarrow G$ defined by:

$$f^{\underline{G}}(x) = \varphi_x(f),$$

where $\varphi_x : E \rightarrow G$ is the homomorphism from \underline{E} into \underline{G} such that $\varphi_x(e) = x$. Therefore:

$$e^{\underline{G}}(x) = x, \quad (fh)^{\underline{G}}(x) = f^{\underline{G}}(x) h^{\underline{G}}(x), \quad (1.4)$$

for any $f, h \in E, x \in G$. (We will usually write $f(x)$ instead of $f^{\underline{G}}(x)$ when we work with a fixed groupoid \underline{G} .)

The following statement is clear.

Proposition 1.2. If \underline{G} is a groupoid and $a \in G$, then $\{f(a) \mid f \in E\}$ is the subgroupoid of \underline{G} generated by a . \square

In the following sections we will use a subset D of E defined as follows:

$$D = \{e^n \mid n \in \mathbf{N}\}, \quad (1.5)$$

where \mathbf{N} is the set of positive integers and

$$e^1 = e, \quad e^{k+1} = e^k e. \quad (1.6)$$

The fact that \underline{F} is injective implies that \underline{F} has the following property:

$$(\forall t, u \in F, m, n \in \mathbf{N})(t^{m+1} = u^{n+1} \Rightarrow t = u \ \& \ m = n). \quad (1.7)$$

¹The notions as subgroupoid, semigroup, variety of groupoids ... have usual meanings.

If \underline{G} is a groupoid, and $b \in G$ is such that
 $(\forall c \in G, n \in \mathbf{N})(n \geq 2 \Rightarrow b \neq c^n)$, (1.8)
then we say that b is a *base element* (or, shortly: a *base*) in \underline{G} .

2. Monoassociative groupoids

We say that a groupoid $\underline{G} = (G, \cdot)$ is *monoassociative* iff, for any $a \in G$, the subgroupoid \underline{Q} of \underline{G} generated by a is associative, i.e. a subsemigroup of \underline{G} . (The class of monoassociative groupoids will be usually denoted by *Mass*.)

The proofs of the following statements are obvious corollaries from the definition of *Mass*.

Proposition 2.1. $\underline{G} \in Mass$ iff for any $f \in E$ and $x \in G$, the following equation
 $f(x) = x^{|f|}$ (2.1)
holds in \underline{G} . \square

Proposition 2.2. If $\underline{G} \in Mass$, then

$$x^m x^n = x^{m+n}, \quad (x^m)^n = x^{mn}, \quad (2.2)$$

for any $x \in G, m, n \in \mathbf{N}$. \square

Proposition 2.3. If \underline{G} is a groupoid, then the following statements are equivalent:

- (a) $\underline{G} \in Mass$.
- (b) \underline{G} is a union of subsemigroups of \underline{G} .
- (c) \underline{G} is a union of cyclic subsemigroups of \underline{G} . \square

Proposition 2.4. *Mass* is a variety of groupoids and:

$$\{f(x) = x^{|f|} \mid f \in E\} \quad (2.3)$$

is an axiom system for this variety. \square

3. Free monoassociative groupoids

Assuming that B is a nonempty set, and \underline{F} an absolutely free groupoid with the basis B , we are looking for a groupoid $\underline{R} = (R, *)$ with the following properties:

- (i) $B \subset R \subset F$;
- (ii) $t \in R \Rightarrow P(t) \subseteq R$;
- (iii) $t, u, tu \in R \Rightarrow t * u = tu$;
- (iv) \underline{R} is a free groupoid in *Mass* with the basis B .

Proposition 2.1 suggests the following set R as a candidate for the carrier of the desired groupoid \underline{R} :

$$R = \{t \in F \mid (\forall f \in E \setminus D, x \in F) f(x) \notin P(t)\}. \quad (3.1)$$

The following properties of R are obvious corollaries of (3.1).

Proposition 3.1. (a) R satisfies (i) and (ii).

(b) $t \in F$ & $m, n \in \mathbf{N}, n \geq 2 \Rightarrow t^m t^n \notin R$.

(c) $t \in F$ & $m, n \in \mathbf{N}, m \geq 2, n \geq 2 \Rightarrow (t^m)^n \notin R$.

(d) $\{t, u\} \subseteq R$ & $tu \notin R \Rightarrow (\exists \alpha \in R, m \geq 1, n \geq 2) tu = \alpha^m \alpha^n$. \square

Now we will describe conditions under which $t^n \in R$.

Proposition 3.2. If $t \in F$ and $n \geq 2$, then:

$$t^n \in R \iff t \in R \text{ \& } t \text{ is a base in } \underline{F}.$$

Proof. Assume that $t \in R$ and t is a base in \underline{F} . By Proposition 3.1 (d), $t^2 \in R$. Assuming that $t^k \in R$, also by Proposition 3.1 (d), we obtain $t^{k+1} = t^k t \in R$.

Conversely, $t^n \in R$, by Proposition 3.1 (a), (d), implies: $t \in R$ and t is a base element in \underline{F} . \square

Now we define an operation $*$ on R , as follows. If $t, u \in R$, then:

$$t * u = \begin{cases} tu, & tu \in R, \\ \alpha^{m+n} & tu = \alpha^m \alpha^n, \quad m, n \in \mathbf{N}, n \geq 2. \end{cases} \quad (3.2)$$

Proposition 3.3. $\underline{R} = (R, *)$ is a groupoid which satisfies the conditions (iii) and (iv).

Proof. 1) By (3.2) and Proposition 3.2, \underline{R} is a groupoid that B is the set of primes in \underline{R} , and the least generating subset of \underline{R} , as well. Moreover, we have:

$$|t * u| = |t| + |u| = |tu|, \quad (3.3)$$

$$\text{cn}(t * u) = \text{cn}(t) \cup \text{cn}(u), \quad (3.4)$$

for any $t, u \in R$.

2) If $t \in R$, $n \in \mathbf{N}$, $f \in E$, then t_*^n , $f_*(t)$ are defined as follows:

$$t_*^1 = t, \quad t_*^{n+1} = t_*^n * t, \quad (3.5)$$

$$e_*(t) = t, \quad (f_1 f_2)_*(t) = (f_{1*}(t)) * (f_{2*}(t)). \quad (3.6)$$

By (3.2), (3.5), (3.6) and Proposition 3.2, we obtain that for any $t \in R$ is a base in \underline{E} , and any $m, n \in \mathbf{N}$, $f \in E$, the following equations hold:

$$t_*^n = t^n, \quad f_*(t) = t^{|f|}, \quad (3.7)$$

$$(t^m)_*^n = t^{mn}, \quad f_*(t^m) = t^{m|f|}. \quad (3.8)$$

Finally, from (3.7) and (3.8), by Proposition 2.1, we obtain that $\underline{R} \in \text{Mass}$.

3) It remains to show that \underline{R} is free in Mass with the basis B .

Let $\underline{G} \in \text{Mass}$, $\lambda : B \rightarrow G$, and φ be the homomorphism from \underline{E} into \underline{G} , which extends λ . Then, for any $t, u \in R$, we have:

$$\varphi(t * u) = \begin{cases} \varphi(tu) = \varphi(t)\varphi(u), & tu \in R, \\ \varphi(\alpha^{m+n}) = \varphi(\alpha)^m \varphi(\alpha)^n = \varphi(t)\varphi(u), & tu = \alpha^m \alpha^n, m, n \in \mathbf{N}, n \geq 2. \end{cases}$$

and this implies that the restriction $\psi = \varphi|_R$ of φ on R is a homomorphism from \underline{R} into \underline{G} , which extends λ . \square

The following properties of \underline{R} can be also easily shown.

Proposition 3.4. If $t \in R$, then t is a base element in \underline{R} iff t is a base element in \underline{E} . \square

Proposition 3.5. If $u \in R$, then there exists a unique pair $(t, k) \in R \times \mathbf{N}$ such that t is a base in \underline{R} and $u = t_*^k (= t^k)$. \square

We say that t is the *base*, and k is the *exponent* of u in \underline{R} . In the case $k \geq 2$, the equation $u = v * w$ holds in \underline{R} iff $v = t^r$, $w = t^s$, and $r + s = k$.

Proposition 3.6. If $u \in R$ is a base element and $u \in R \setminus B$, then there is a unique pair $(v, w) \in R^2$ such that $u = v * w (= vw)$; moreover, v and w have different bases. \square

Proposition 3.7. If $t, u, v \in R$, then:

(a) $t * u = u * t$ iff t and u have the same base.

(b) $(t * u) * v = t * (u * v)$ iff t, u , and v have the same base. \square

Proposition 3.8. If $B = \{b\}$ is a one element set, then $R = \{b^n \mid n \geq 1\}$, and $b^m * b^n = b^{m+n}$. (Therefore, \underline{R} is isomorphic with the additive semigroup of positive integers.) \square

4. Injective objects in the variety of monoassociative groupoids

Looking for a convenient class of "injective groupoids" in a variety \mathcal{V} of groupoids we choose as axioms of such a class corresponding properties of free objects in \mathcal{V} that are "near" the statement (1.1). In the case of Mass , such statements are Proposition 3.5 and Proposition 3.6, and that is why we give the following definition.

We say that a groupoid $H \in \text{Mass}$ is *injective* in Mass , i.e. it is in Massin , iff it satisfies the following conditions:

(i) For any $a \in H$ there is a unique pair $(b, k) \in H \times \mathbf{N}$ such that $a = b_*^k$ and b is a base in \underline{H} . (We say that b is the *base* and k is the *exponent* of a in \underline{H} , and write $b = \beta(a)$, $k = \varepsilon(a)$.)

(ii) Let $a \in H$ be not prime in \underline{H} .

(ii.1) If $b = \beta(a)$ and $\varepsilon(a) \geq 2$, then

$$a = cd \Rightarrow \beta(c) = \beta(d) = b \ \& \ \varepsilon(c) + \varepsilon(d) = \varepsilon(a).$$

(ii.2) If $c, d \in H$ are such that $\beta(c) \neq \beta(d)$, then $\beta(cd) = cd$, and: $cd = c'd' \Rightarrow (c, d) = (c', d')$.

As corollaries of the given definition and Propositions 3.5–3.7, we obtain the following properties of Massin .

Proposition 4.1. The class of free groupoids in Mass (shortly: Massfr) is a subclass of Massin . \square

Proposition 4.2. A groupoid $\underline{H} \in \text{Massin}$ contains only one base element iff \underline{H} is isomorphic to the additive semigroup of positive integers. \square

Proposition 4.3. Each $\underline{H} \in \text{Massin}$ is infinite. \square

Proposition 4.4. Every groupoid $\underline{H} \in \text{Massin}$ contains infinitely many subgroupoids that are not injective.

Namely, if b is a base in \underline{H} , then for any $i \geq 2$, $Q_i = \{b^n \mid n \geq i\}$ is a subgroupoid of \underline{H} and $Q_i \notin \text{Massin}$. \square

Proposition 4.5. *Massfr* is a proper subclass of *Massin*.

Proof. Let A be an infinite set and let $H = A \times \mathbf{N}$. Instead of $(a, n) \in H$ we will write a^n , and moreover, a instead of a^1 . The fact that A is infinite implies that A , H and

$$C = \{(a^m, b^n) \mid a, b \in A, a \neq b, m, n \in \mathbf{N}\},$$

have the same cardinality. Let $\varphi : C \rightarrow H$ be an injective mapping and define a groupoid $\underline{H} = (H, \bullet)$ as follows:

$$(\forall a, b \in A, a \neq b, m, n \in \mathbf{N}) \quad a^m \bullet a^n = a^{m+n}, \quad a^m \bullet b^n = \varphi(a^m, b^n).$$

Then $\underline{H} \in \text{Massin}$.

Namely, $a = \beta(a^k)$, $k = \varepsilon(a^k)$, for each $a \in A$, $k \in \mathbf{N}$. And, if $a^m, b^n \in H$, $a \neq b$, then $a^m \bullet b^n = \varphi(a^m, b^n)$ is a base that is not prime in \underline{H} . The injectiveness of φ implies that the condition (ii) of the definition holds as well. Then, $H \setminus \text{im}(\varphi)$ is the set of primes in \underline{H} . Therefore, if φ is bijective, then the set of primes in \underline{H} is empty, and then $\underline{H} \notin \text{Massfr}$. \square

Proposition 4.6. If $\underline{H} \in \text{Massin}$ is such that there exist at least two distinct base elements in \underline{H} , then the set of base elements in \underline{H} is infinite.

Proof. Let b, c be base elements in \underline{H} and $b \neq c$. Then, $\{b^k c \mid k \geq 1\}$ is an infinite set of base elements in \underline{H} . \square

As a corollary we obtain the following.

Proposition 4.7. If $\underline{H} \in \text{Massin}$, then the following conditions are equivalent:

- (a) \underline{H} is commutative;
- (b) \underline{H} is associative;
- (c) \underline{H} is isomorphic to the additive semigroup of positive integers;
- (d) There is only one base element in \underline{H} ;
- (e) $\underline{H} \in \text{Massfr}$ with one-element basis. \square

Below we assume that $\underline{H} \in \text{Massin}$, \underline{Q} is a subgroupoid of \underline{H} and the following notation:

$$\begin{aligned} \beta(H) &= \{\beta(a) \mid a \in H\}, & C &= Q \cap \beta(H), \\ D &= \{b \in \beta(H) \setminus Q \mid (\exists a \in Q) b = \beta(a)\}, \\ r_b &= \min\{k \mid b^k \in Q\}, & \text{where } b &\in D. \end{aligned}$$

Proposition 4.8. If $D = \emptyset$, then $\underline{Q} \in \text{Massin}$.

Proof. This is a consequence from the definition of *Massin*. \square

Proposition 4.9. If $D \neq \emptyset$, then the following statements are true.

- 1) For every $b \in D$, the element b^{r_b} is prime in \underline{Q} .
- 2) If, for every $b \in D$, $b^s \in Q$ implies $r_b \mid s$, then $\underline{Q} \in \text{Massin}$.
- 3) If there are $b \in D$ and $s \in \mathbf{N}$ such that r_b does not divide s and $b^s \in Q$, and if s is the least integer with this property, then b^s is prime in \underline{Q} and $\underline{Q} \notin \text{Massin}$.

Proof. 1) If b^r ($r = r_b$) were not prime in \underline{Q} , then we would have $b^r = b^i b^j$ for some $b^i, b^j \in Q$, $i + j = r$, and this contradicts the choice of r .

2) Suppose that $a \in Q$ is such that $b = \beta(a) \in D$. By 1), b^r ($r = r_b$) is the base of a in \underline{Q} and the exponent of a in \underline{Q} is $\varepsilon(a) \mid r$. Thus $\underline{Q} \in \text{Massin}$.

3) Let $s = \min\{k \in \mathbf{N} \mid b^k \in Q \text{ and } r \text{ does not divide } k\}$. Then b^s is prime in \underline{Q} . (Namely, if b^s were not prime, then we would have $b^s = b^i b^j$ for some $b^i, b^j \in Q$, $(i + j = s)$. By 1), $r \mid i$ and $r \mid j$, which implies $r \mid s$, a contradiction with the choice of s .) Thus the elements b^r, b^s are prime in \underline{Q} . Since $(b^r)^s = b^{r+s} = (b^s)^r$, we have that b^{r+s} has two distinct bases in \underline{Q} , and thus $\underline{Q} \notin \text{Massin}$. \square

As a corollary of Propositions 4.8–4.9, we obtain.

Proposition 4.10. $\underline{Q} \notin \text{Massin}$ iff there is $b \in \beta(H)$ and $r, s \in \mathbf{N}$ such that $2 \leq r < s$ and b^r, b^s are prime in \underline{Q} . \square

5. Bruck Theorem for the variety of monoassociative groupoids

Below we show the following proposition, analogous to Proposition 1.1, that we call **Bruck Theorem** for the variety of monoassociative groupoids ([4]).

Proposition 5.1. A groupoid $\underline{H} \in \text{Mass}$ is free in Mass iff the following two conditions are satisfied:

(a) $\underline{H} \in \text{Massin}$.

(b) The set B of primes in \underline{H} generates \underline{H} .

Proof. If $\underline{H} \in \text{Massfr}$ then, by Proposition 4.5, $\underline{H} \in \text{Massin}$, and, by Proposition 3.3, the set B of primes generates \underline{H} .

Let $\underline{H} \in \text{Massin}$ and the set B of primes generates \underline{H} .

If $B = \{b\}$, then $H = \{b^n \mid n \geq 1\}$, and b is the unique base element in \underline{H} and, by Proposition 4.2, \underline{H} is free in Mass with the basis $\{b\}$.

It remains the case when B contains at least two distinct elements. As in §4 we denote by $\beta(H)$ the set of bases in \underline{H} . Clearly, each prime in \underline{H} belongs to $\beta(H)$, and thus $B = B_0 \subseteq \beta(H)$. By (ii) of the definition of injectiveness, we also have $B_1 \subseteq \beta(H)$, where

$$B_1 = \{a^m b^n \mid a, b \in B_0, a \neq b, m, n \in \mathbf{N}\}.$$

Assume that: B_0, B_1, \dots, B_k are nonempty sets of bases such that $B_i \cap B_j = \emptyset$ if $i \neq j$. Define B_{k+1} by:

$$B_{k+1} = \{c^m d^n \mid m, n \in \mathbf{N}, c \neq d, \{c, d\} \subseteq B_0 \cup \dots \cup B_k, \{c, d\} \cap B_k \neq \emptyset\}.$$

By (ii) of the definition, we have $B_{k+1} \subseteq \beta(H)$, $B_{k+1} \neq \emptyset$ and $B_{k+1} \cap B_i = \emptyset$, for each $i \in \{1, 2, \dots, k\}$. Moreover, the fact that $B (= B_0)$ generates \underline{H} implies that

$$\beta(H) = \cup \{B_s \mid s \geq 0\}.$$

If

$$B_i^\wedge = \{\alpha^s \mid \alpha \in B_i, s \in \mathbf{N}\},$$

then $i \neq j$ implies $B_i^\wedge \cap B_j^\wedge = \emptyset$ and

$$H = \cup \{B_i^\wedge \mid i \geq 1\}.$$

Let $\underline{G} \in \text{Mass}$ and $\lambda : B \rightarrow G$. Define a sequence of mappings $\varphi_i : B_i^\wedge \rightarrow G$ as follows:

$$b \in B_0, n \geq 1 \Rightarrow \varphi_0(b^n) = (\lambda(b))^n;$$

$$c^m d^n \in B_1, n \geq 1 \Rightarrow \varphi_1((c^m d^n)^s) = ((\varphi_0(c))^m (\varphi_0(d))^n)^s;$$

$$c^m d^n \in B_{k+1}, c \in B_i, d \in B_j \Rightarrow \varphi_{k+1}((c^m d^n)^s) = ((\varphi_i(c))^m (\varphi_j(d))^n)^s.$$

Then, the union $\varphi = \cup_{k=0}^\infty \varphi_k$ is a homomorphism of \underline{H} into \underline{G} that extends the given mapping $\lambda : B \rightarrow G$. \square

Below we assume that $\underline{H} = (H, \cdot) \in \text{Massfr}$, \underline{Q} is a subgroupoid of \underline{H}

Below we assume that $\underline{H} = (H, \cdot) \in \text{Massfr}$, \underline{Q} is a subgroupoid of \underline{H} and B is the set of primes (i.e. B is the basis) of \underline{H} .

Using the fact that any groupoid $\underline{H} = (H, \cdot) \in \text{Massfr}$ with the basis B is isomorphic with the groupoid \underline{R} constructed in §3, and the statements (3.3) and (3.4), we can state the following

Proposition 5.2. There exist a mapping $x \mapsto |x|$ of H into \mathbf{N} , and a mapping $x \mapsto \text{cn}(x)$ of H into the set L_B of all finite nonempty subsets of B , such that

$$1) \quad |b| = 1, \quad |xy| = |x| + |y|,$$

$$2) \quad \text{cn}(b) = \{b\}, \quad \text{cn}(xy) = \text{cn}(x) \cup \text{cn}(y),$$

for any $b \in B, x, y \in H$. \square ²

²Note that the existence of such mappings can be shown without using the free groupoid \underline{H} . Namely, the fact that $(\mathbf{N}, +) \in \text{Mass}$ implies that there exists a homomorphism $|\cdot| : H \rightarrow \mathbf{N}$ such that $|b| = 1$ for each $b \in B$. Also, the fact that $(L_B, \cup) \in \text{Mass}$ implies that there is a homomorphism $\text{cn} : H \rightarrow L_B$, such that $\text{cn}(b) = \{b\}$ for each $b \in B$.

Proposition 5.3. The set P of primes in \underline{Q} is nonempty and generates \underline{Q} .

Proof. Assume that $p \in Q$ is such that

$$|p| = \min \{|x| \mid x \in Q\}.$$

Then p is a prime in \underline{Q} , and thus the set P of primes in \underline{Q} is nonempty.

Denote by \underline{T} the subgroupoid of \underline{Q} generated by P and assume that for each $a \in Q$ such that $|a| \leq k$, we have $a \in T$. (In the case $|a| = 1$, we have $a \in P$.) Then, if $d \in Q$ is such that $|d| = k + 1$, we have: $d \in T$ if $d \in P$, and if $d \in Q \setminus P$, then there exist $b, c \in Q$ such that $d = bc$. Then, by Proposition 5.2.1), $|b|, |c| \leq k$, and therefore $b, c \in T$, which implies that $d \in T$. \square

As a corollary of Propositions 4.8-4.9, Proposition 5.1. and Proposition 5.3, we obtain the following characterization of free subgroupoids of groupoids in Massfr .

Proposition 5.4. If $\underline{H} \in Massfr$ and \underline{Q} is a subgroupoid of \underline{H} , then the following conditions are equivalent:

- (a) $\underline{Q} \in Massin$;
- (b) $\underline{Q} \in Massfr$;
- (c) There are no prime elements b^r, b^s in \underline{Q} , where b is a base in \underline{H} and $2 \leq r < s$. \square

A corollary of Proposition 4.2 is the following

Proposition 5.5. If $\underline{H} \in Massfr$ is with one-element basis and \underline{Q} is a subgroupoid of \underline{H} , then: $\underline{Q} \in Massfr$ iff \underline{Q} is cyclic. \square

Proposition 5.6. Let $\underline{H} \in Massfr$ with the two-element basis $B = \{a, b\}$ and \underline{Q} be the subgroupoid of \underline{H} generated by

$$C = \{a^k b^k \mid k \in \mathbb{N}\}.$$

Then $\underline{Q} \in Massfr$ with the infinite basis C .

Proof. The assumption $a \neq b$ implies that each element $c \in C$ is a base in \underline{H} ; moreover, $a^m b^m = a^n b^n$ implies $m = n$, i.e. the set C is infinite.

Note that, by (3.4), $(\forall t \in \underline{Q})(cn(t) = \{a, b\})$, and thus $a^k, b^k \notin \underline{Q}$. Therefore, every $c \in C$ is prime in \underline{Q} and, by Proposition 5.4 (c), $\underline{Q} \in Massfr$. \square

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ЗА МОНОАСОЦИЈАТИВНИТЕ ГРУПОИДИ

Резиме

Предмет на оваа работа е многуобразието (означено со $Mass$) од моноасоцијативни групоиди, т.е. групоиди во кои секој цикличен подгрупоид е полугрупа. Даден е опис на слободните објекти во $Mass$. Користејќи соодветна дефиниција на поимот инјективен групоид во $Mass$, се покажува дека еден групоид \underline{H} е слободен во $Mass$ ако и само ако \underline{H} е инјективен во $Mass$ и множеството прости елементи во \underline{H} го генерира \underline{H} . (Ова својство е наречено Теорема на Браќ за $Mass$.) Ниедна од класите $Massin$ (т.е. класата инјективни објекти во $Mass$) и $Massfr$ (т.е. класата слободни објекти во $Mass$) не е наследна. Добиена е карактеризација на слободните подгрупоиди од еден групоид $\underline{H} \in Massfr$ и покажано е дека секој групоид $\underline{H} \in Mass$ со двоелементна база има подгрупоид $\underline{Q} \in Massfr$ со бесконечна база.