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GENERALISATIONS OF STEFFENSEN'S INEQUALITY BY HERMITE'S POLYNOMIAL

JULIJE JAKŠETIĆ, JOSIP PEČARIĆ, AND ANAMARIJA PERUŠIĆ

Abstract. We study generalizations of Steffensen's inequality using Hermite expansions with integral reminder. In comparing differences of two weighted integrals we vary on the number of knots in expansion which leads us to generalization of conditions for Steffensen's inequality. After that, we construct exponentially convex functions and Cauchy means.

1. Introduction

Let $-\infty < \alpha < b < \infty$, and $a \leq a_1 < a_2 ... < a_r \leq b$, $(r \geq 2)$ be given. For $f \in C^n[a,b]$ a unique polynomial $P_H(t)$ of degree $(n-1)$, exists, fulfilling one of the following conditions:

Hermite conditions:

$$P_H^{(i)}(a_j) = f^{(i)}(a_j);\ 0 \leq i \leq k_j,\ 1 \leq j \leq r,\ \sum_{j=1}^{r} k_j + r = n,$$

in particular:

Simple Hermite or osculatory conditions: $(n = 2m,\ r = m,\ k_j = 1$ for all $j)$

$$P_O(a_j) = f(a_j),\ P_O'(a_j) = f'(a_j),\ 1 \leq j \leq m,$$

Lagrange conditions: $(r = n,\ k_j = 0$ for all $j)$

$$P_L(a_j) = f(a_j),\ 1 \leq j \leq n,$$

Type $(m, n-m)$ **conditions:** $(r = 2,\ 1 \leq m \leq n-1,\ k_1 = m-1,\ k_2 = n-m-1)$

$$P_{mn}^{(i)}(a) = f^{(i)}(a),\ \ 0 \leq i \leq m-1,$$
$$P_{mn}^{(i)}(b) = f^{(i)}(b),\ \ 0 \leq i \leq n-m-1,$$

One-point Taylor conditions: $(r = 1, k_1 = n-1)$

$$P_T^{(i)}(a) = f^{(i)}(a),\ \ 0 \leq i \leq n-1.$$

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Two-point Taylor conditions: $(n = 2m,\ r = 2,\ k_1 = k_2 = m - 1)$

$$P_{2T}^{(i)}(a) = f^{(i)}(a),\ P_{2T}^{(i)}(b) = f^{(i)}(b),\ \ 0 \le i \le m-1.$$

The associated error $e_H(t)$ can be represented in terms of the Green's function $G_H(t,s)$ for the multipoint boundary value problem

$$z^{(n)}(t) = 0,\ z^{(i)}(a_j) = 0,\ 0 \le i \le k_j,\ 1 \le j \le r,$$

that is, the following result holds (see [1]):

Theorem 1. *Let $F \in C^n[a,b]$, and let P_H be its Hermite interpolating polynomial. Then*

$$\begin{aligned} F(t) &= P_H(t) + e_H(t) \\ &= \sum_{j=1}^{r} \sum_{i=0}^{k_j} H_{ij}(t) F^{(i)}(a_j) + \int_a^b G_H(t,s) F^{(n)}(s) ds, \end{aligned} \tag{1.1}$$

where H_{ij} are fundamental polynomials of the Hermite basis defined by

$$H_{ij}(t) = \frac{1}{i!} \frac{\omega(t)}{(t-a_j)^{k_j+1-i}} \sum_{k=0}^{k_j-i} \frac{1}{k!} \frac{d^k}{dt^k} \left(\frac{(t-a_j)^{k_j+1}}{\omega(t)} \right) \Big|_{t=a_j} (t-a_j)^k, \tag{1.2}$$

where

$$\omega(t) = \prod_{j=1}^{r} (t-a_j)^{k_j+1}, \tag{1.3}$$

and G_H is the Green's function defined by

$$G_H(t,s) = \begin{cases} \sum\limits_{j=1}^{\ell} \sum\limits_{i=0}^{k_j} \frac{(a_j-s)^{n-i-1}}{(n-i-1)!} H_{ij}(t), & s \le t \\ -\sum\limits_{j=\ell+1}^{r} \sum\limits_{i=0}^{k_j} \frac{(a_j-s)^{n-i-1}}{(n-i-1)!} H_{ij}(t), & s \ge t. \end{cases} \tag{1.4}$$

for all $a_\ell \le s \le a_{\ell+1}, \ell = 0,1,\ldots,r$ $(a_0 = a,\ a_{r+1} = b)$.

Remark 1.1: In particular case, for one-point Taylor conditions

$$H_{i1}(t) = \frac{(t-a)^i}{i!}, \qquad i = 0,1,\ldots,n-1,$$

and Green's function G_T is

$$G(t,s) = \begin{cases} \frac{(t-s)^{n-1}}{(n-1)!}, & s \le t; \\ 0, & s > t, \end{cases}$$

so Theorem 1 gives us classical Taylor theorem with integral reminder:

$$F(t) = \sum_{i=0}^{n-1} (t-a)^i \frac{F^{(i)}(a)}{i!} + \int_a^t (t-s)^{n-1} \frac{F^{(n)}(s)}{n!} ds.$$

For two-point Taylor conditions, $i = 0, 1, \dots, m-1$

$$H_{i1}(t) = \sum_{k=0}^{m-1-i} \binom{m+k-1}{k} \frac{(t-a)^i}{i!} \left(\frac{t-b}{a-b}\right)^m \left(\frac{t-a}{b-a}\right)^k$$

$$H_{i2}(t) = \sum_{k=0}^{m-1-i} \binom{m+k-1}{k} \frac{(t-b)^i}{i!} \left(\frac{t-a}{b-a}\right)^m \left(\frac{x-b}{a-b}\right)^k$$

and Green's function G_{2T} is

$$G_{2T}(t,s) = \begin{cases} \frac{(-1)^m}{(2m-1)!} p^m(t,s) \sum_{j=0}^{m-1} \binom{m-1+j}{j} (t-s)^{m-1-j} q^j(t,s), & s \le t \\ \frac{(-1)^m}{(2m-1)!} q^m(t,s) \sum_{j=0}^{m-1} \binom{m-1+j}{j} (s-t)^{m-1-j} p^j(t,s), & s \ge t \end{cases} \tag{1.5}$$

and

$$p(t,s) = \frac{(s-a)(b-t)}{b-a}, \quad q(t,s) = p(s,t), \ \forall t, s \in [a,b].$$

The following lemma describes positivity of Green's function (1.4) (see Beesack [3] and Levin [5]).

Lemma 1. *The Green's function $G_H(t,s)$ has the following properties:*

(i) $\dfrac{G_H(t,s)}{\omega(t)} > 0, \quad a_1 \le t \le a_r, \ a_1 < s < a_r;$

(ii) $G_H(t,s) \le \dfrac{1}{(n-1)!(b-a)} |\omega(t)|;$

(iii) $\displaystyle\int_a^b G_H(t,s)ds = \frac{\omega(t)}{n!}.$

2. Difference of integrals and Steffensen inequality

If $[a,b] \cap [c,d] \neq \emptyset$ we have four possible cases for two intervals $[a,b]$ and $[c,d]$. First case is $[c,d] \subset [a,b]$, second case is $[a,b] \cap [c,d] = [c,b]$ and other two cases are obtained by changing $a \leftrightarrow c$, $b \leftrightarrow d$. Hence, in the following theorem we will only observe first two cases.

In this paper by $T_{w,n}^{[a,b],H^1}$ and $T_{u,n}^{[c,d],H^2}$ we will denote

$$T_{w,n}^{[a,b],H^1} = \sum_{j=1}^{r} \sum_{i=0}^{k_j} f^{(i)}(a_j) \int_a^b w(t) H_{ij}^1(t) dt$$

$$T_{u,n}^{[c,d],H^2} = \sum_{j=1}^{s} \sum_{i=0}^{k_j} f^{(i)}(c_j) \int_c^d u(t) H_{ij}^2(t) dt$$

where H^1 and H^2 concern Hermite basis for knots

$-\infty < a \le a_1 < a_2 \dots < a_{r_1} \le b < \infty$ and
$-\infty < c \le c_1 < c_2 \dots < c_{r_2} \le d < \infty$, respectively.

Theorem 2. *Let $f : [a,b] \cup [c,d] \to \mathbb{R}$ be of class $C^{(n)}$ on $[a,b] \cup [c,d]$ for some $n \geq 1$. Let $w : [a,b] \to [0,\infty\rangle$ and $u : [c,d] \to [0,\infty\rangle$. Then if $[a,b] \cap [c,d] \neq \emptyset$ we have*

$$\begin{aligned} \int_a^b w(t) f(t)\, dt - \int_c^d u(t) f(t)\, dt - T_{w,n}^{[a,b],H^1} + T_{u,n}^{[c,d],H^2} = \\ = \int_a^{\max\{b,d\}} K_n(s) f^{(n)}(s)\, ds, \end{aligned} \tag{2.1}$$

where in case $[c,d] \subseteq [a,b]$,

$$K_n(s) = \begin{cases} \int_a^b w(t) G_{H^1}(t,s)\, dt, & s \in [a,c], \\ \int_a^b w(t) G_{H^1}(t,s)\, dt - \int_c^d u(t) G_{H^2}(t,s)\, dt, & s \in \langle c,d], \\ \int_a^b w(t) G_{H^1}(t,s)\, dt, & s \in \langle d,b], \end{cases} \tag{2.2}$$

and in case $[a,b] \cap [c,d] = [c,b]$,

$$K_n(s) = \begin{cases} \int_a^b w(t) G_{H^1}(t,s)\, dt & s \in [a,c], \\ \int_a^b w(t) G_{H^1}(t,s)\, dt - \int_c^d u(t) G_{H^2}(t,s)\, dt, & s \in \langle c,b], \\ -\int_c^d u(t) G_{H^2}(t,s)\, dt, & s \in \langle b,d]. \end{cases} \tag{2.3}$$

Proof. We use Theorem 1 to express the function f first on knots

$$-\infty < a \leq a_1 < a_2 < \cdots < a_{r_1} \leq b < \infty \text{ and then on}$$
$$-\infty < c \leq c_1 < c_2 \ldots < c_{r_2} \leq d < \infty.$$

We multiply both sides with functions w and u respectively, and then integrate both sides. By substraction and use of Fubini theorem we get desired result. □

Using Theorem 2 we get, in particular, Steffensen inequality (see [7]).

Corollary 2.1. *Suppose that f is increasing and w is integrable on $[a,b]$ with $0 \leq w \leq 1$ and*

$$\lambda = \int_a^b w(t) dt. \tag{2.4}$$

Then we have

$$\int_{b-\lambda}^b f(t) dt \geq \int_a^b w(t) f(t) dt \geq \int_a^{a+\lambda} f(t) dt. \tag{2.5}$$

Proof. 1° We first prove $\int_a^b f(t) w(t) dt \geq \int_a^{a+\lambda} f(t) dt$.

For Hermite polynomials H^1 and H^2 we consider one-point Taylor conditions on

$[a,b]$ and $[a,a+\lambda]$ respectively. Then from

$$K_1(s) = \begin{cases} \int_s^b w(t)dt - (a+\lambda) + s, & s \in [a, a+\lambda], \\ \int_s^b w(t)dt, & s \in \langle d, a+\lambda], \end{cases} \tag{2.6}$$

it follows $K_1(s) \geq 0$. Now (2.1) give us

$$\int_a^b w(t)f(t)dt - \int_a^{a+\lambda} f(t)dt - f(a)\lambda + f(a)\lambda = \int_a^b K_1(s)f'(s)ds \geq 0,$$

concluding that $\int_a^b w(t)f(t)dt - \int_a^{a+\lambda} f(t)dt \geq 0$.

2° Now we prove $\int_{b-\lambda}^b f(t)dt \geq \int_a^b w(t)f(t)dt$.

For Hermite polynomials H^1 and H^2 here we consider one-point Taylor conditions on $[a,b]$ and $[b-\lambda,b]$ respectively. Then

$$K_1(s) = \begin{cases} \int_s^b w(t)dt, & s \in [a, b-\lambda], \\ \int_s^b (w(t)-1)dt, & s \in \langle b-\lambda, b], \end{cases} \tag{2.7}$$

Now (2.1) give us

$$\int_a^b w(t)f(t)dt - \int_{b-\lambda}^b f(t)dt - f(a)\lambda + f(b-\lambda)\lambda = \int_a^b K_1(s)f'(s)ds \leq$$
$$\leq \lambda \int_a^{b-\lambda} f'(s)ds,$$

concluding that $\int_a^b w(t)f(t)dt - \int_{b-\lambda}^b f(t)dt \leq 0$. □

Theorem 3. *Let $f : [a,b] \cup [c,d] \to \mathbb{R}$ be $n-$convex on $[a,b] \cup [c,d]$ and let $w : [a,b] \to [0,\infty\rangle$ and $u : [c,d] \to [0,\infty\rangle$. Then if $[a,b] \cap [c,d] \neq \emptyset$ and*

$$K_n(s) \geq 0, \tag{2.8}$$

we have

$$\int_a^b w(t) f(t) dt - T_{w,n}^{[a,b],H^1} \geq \int_c^d u(t) f(t) dt - T_{u,n}^{[c,d],H^2} \tag{2.9}$$

where in case $[c,d] \subseteq [a,b]$, $K_n(s)$ is defined by (2.2) and in case $[a,b] \cap [c,d] = [c,b]$, $K_n(s)$ is defined by (2.3).

Proof. Since f is n-convex, without loss of generality we can assume that f is $n-$times differentiable and $f^{(n)} \geq 0$ see [8, p. 16 and p. 293]. Now we can apply Theorem 2 to obtain (2.9). □

Remark 2.1: It is easy to find kernels K_n such that (2.8) is fulfilled. For example, if we take $a \leq a_1 < a_2 ... < a_{r_1} \leq b$ and all $k_1, \ldots, k_{r_1}$ are odd ($\sum_{j=1}^{r_1} k_j + r_1 = n$) then $\omega_1(t) = \prod_{j=1}^{r_1}(t-a_j)^{k_j+1} \geq 0$ and according (i)-part of Lemma 1 $G_{H^1}(t,s) \geq 0$. Similarly, if we take $c \leq c_1 < c_2 ... < c_{r_2} = d < \infty$, all $m_1, \ldots, m_{r_2-1}$ are odd and m_{r_2} is even ($\sum_{j=1}^{r_2} m_j + r_2 = n$), then $\omega_2(t) = \prod_{j=1}^{r_2}(t-a_j)^{m_j+1} \leq 0$ and again, according (i)-part of Lemma 1, $G_{H^2}(t,s) \leq 0$.
Particularly, in one-point Taylor case this is valid for any $n \in \mathbb{N}$ and in two-point Taylor case this is valid for any even $m \in \mathbb{N}$.

3. Generalization of Steffensen's inequality by Hermite's polynomial

The well-known Steffensen inequality is given and proved by J.F. Steffensen in 1918 in paper [7].

In this section we use Hermite expansion in order to generalize Steffensen inequality. For special choice of weights and intervals from previous section we obtain generalization of Steffensen's inequality.

Theorem 4. *Let $f:[a,b]\cup[a,a+\lambda]\to\mathbb{R}$ be $n-$convex on $[a,b]\cup[a,a+\lambda]$ and let $w:[a,b]\to[0,\infty\rangle$. Then if*

$$K_n(s)\geq 0, \tag{3.1}$$

we have

$$\int_a^b w(t)\,f(t)\,dt - T_{w,n}^{[a,b],H^1} \geq \int_a^{a+\lambda} f(t)\,dt - T_{1,n}^{[a,a+\lambda],H^2} \tag{3.2}$$

where in case $a\leq a+\lambda\leq b$,

$$K_n(s)=\begin{cases} \int_a^b w(t)G_{H^1}(s,t)\,dt-\int_a^{a+\lambda}G_{H^2}(s,t)\,dt, & s\in[a,a+\lambda],\\[2ex] \int_a^b w(t)G_{H^1}(s,t)\,dt, & s\in\langle a+\lambda,b], \end{cases} \tag{3.3}$$

and in case $a<b\leq a+\lambda$,

$$K_n(s)=\begin{cases} \int_a^b w(t)G_{H^1}(s,t)\,dt-\int_a^{a+\lambda}G_{H^2}(s,t)\,dt, & s\in[a,b],\\[2ex] -\int_a^{a+\lambda}G_{H^2}(s,t)\,dt, & s\in\langle b,a+\lambda]. \end{cases} \tag{3.4}$$

Proof. We take $c=a$, $d=a+\lambda$ and $u(t)=1$ in Theorem 3. □

Theorem 5. *Let $f:[a,b]\cup[b-\lambda,b]\to\mathbb{R}$ be $n-$convex on $[a,b]\cup[b-\lambda,b]$ and let $w:[a,b]\to[0,\infty\rangle$. Then if*

$$K_n(s)\leq 0, \tag{3.5}$$

we have

$$\int_{b-\lambda}^b f(t)\,dt - T_{1,n}^{[b-\lambda,b],H^2}(x) \geq \int_a^b w(t)\,f(t)\,dt - T_{w,n}^{[a,b],H^1}(x) \tag{3.6}$$

where in case $a\leq b-\lambda\leq b$,

$$K_n(s)=\begin{cases} \int_a^b w(t)G_{H^1}(s,t)\,dt, & s\in[a,b-\lambda],\\[2ex] \int_a^b w(t)G_{H^1}(s,t)\,dt-\int_{b-\lambda}^b G_{H^2}(s,t)\,dt, & s\in\langle b-\lambda,b], \end{cases} \tag{3.7}$$

and in case $b-\lambda\leq a\leq b$,

$$K_n(s)=\begin{cases} -\int_{b-\lambda}^b G_{H^2}(s,t)\,dt, & s\in[b-\lambda,a],\\[2ex] \int_a^b w(t)G_{H^1}(s,t)\,dt-\int_{b-\lambda}^b G_{H^2}(s,t)\,dt, & s\in\langle a,b]. \end{cases} \tag{3.8}$$

4. Estimation of the difference

Theorem 6. *Suppose that all assumptions of Theorem 3 hold. Additionally assume (p,q) is a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$, $1/p + 1/q = 1$. Let $\left|f^{(n)}\right|^p : [a,b] \cup [c,d] \to \mathbb{R}$ be an R-integrable function for some $n \geq 1$. Then we have*

$$\begin{aligned} &\left| \int_a^b w(t) f(t)\, dt - \int_c^d u(t) f(t)\, dt - T_{w,n}^{[a,b],H^1} + T_{u,n}^{[c,d],H^2} \right| \\ &\leq \left\| f^{(n)} \right\|_p \left(\int_a^{\max\{b,d\}} \left| K_n(s) \right|^q ds \right)^{\frac{1}{q}}. \end{aligned} \tag{4.1}$$

The constant $\left(\int_a^{\max\{b,d\}} |K_n(s)|^q\, ds\right)^{1/q}$ in the inequality (4.1) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. Using inequality (2.9) and applying Hölder inequality we obtain

$$\begin{aligned} &\left| \int_a^b w(t) f(t)\, dt - \int_c^d u(t) f(t)\, dt - T_{w,n}^{[a,b],H^1} + T_{u,n}^{[c,d],H^2} \right| \\ &= \left| \int_a^{\max\{b,d\}} K_n(s) f^{(n)}(s) ds \right| \leq \left\| f^{(n)} \right\|_p \left(\int_a^{\max\{b,d\}} \left| K_n(s) \right|^q ds \right)^{\frac{1}{q}}. \end{aligned}$$

For the proof of the sharpness of the constant $\left(\int_a^{\max\{b,d\}} |K_n(s)|^q\, ds\right)^{\frac{1}{q}}$ we will find a function f for which the equality in (4.1) is obtained.
For $1 < p < \infty$ take f to be such that

$$f^{(n)}(s) = \operatorname{sgn} K_n(s) \left| K_n(s) \right|^{\frac{1}{p-1}}.$$

For $p = \infty$ take

$$f^{(n)}(s) = \operatorname{sgn} K_n(s).$$

For $p = 1$ we shall prove that

$$\begin{aligned} &\left| \int_a^{\max\{b,d\}} K_n(s) f^{(n)}(s) ds \right| \leq \\ &\leq \max_{s \in [a, \max\{b,d\}]} \left| K_n(s) \right| \left(\int_a^{\max\{b,d\}} \left| f^{(n)}(s) \right| ds \right) \end{aligned} \tag{4.2}$$

is the best possible inequality. Suppose that $|K_n(s)|$ attains its maximum at $s_0 \in [a, \max\{b,d\}]$. First we assume that $K_n(s_0) > 0$. For ε small enough we define $f_\varepsilon(s)$ by

$$f_\varepsilon(s) = \begin{cases} 0, & a \leq s \leq s_0, \\ \frac{1}{\varepsilon\, n!}(s - s_0)^n, & s_0 \leq s \leq s_0 + \varepsilon, \\ \frac{1}{n!}(s - s_0)^{n-1}, & s_0 + \varepsilon \leq s \leq \max\{b,d\}. \end{cases}$$

Then for ε small enough

$$\left|\int_a^{\max\{b,d\}} K_n(s)f^{(n)}(s)ds\right| = \left|\int_{s_0}^{s_0+\varepsilon} K_n(s)\frac{1}{\varepsilon}ds\right| = \frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} K_n(s)ds.$$

Now from inequality (4.2) we have

$$\frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} K_n(s)ds \leq K_n(s_0)\int_{s_0}^{s_0+\varepsilon}\frac{1}{\varepsilon}ds = K_n(s_0).$$

Since,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} K_n(s)ds = K_n(s_0)$$

the statement follows. In case $K_n(s_0) < 0$ we define

$$f_\varepsilon(s) = \begin{cases} \frac{1}{n!}(s-s_0-\varepsilon)^{n-1},, & a \leq s \leq s_0, \\ -\frac{1}{\varepsilon\, n!}(s-s_0-\varepsilon)^n, & s_0 \leq s \leq s_0+\varepsilon, \\ 0, & s_0+\varepsilon \leq s \leq \max\{b,d\} \end{cases}$$

and the rest of the proof is the same as above. □

Theorem 7. *Suppose that all assumptions of Theorem 4 hold. Additionally assume (p,q) is a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$, $1/p+1/q=1$. Let $\left|f^{(n)}\right|^p : [a,b]\cup[a,a+\lambda] \to \mathbb{R}$ be an R-integrable function for some $n \geq 1$. Let $K_n(s)$ be defined by* (3.3) *in case $a \leq a+\lambda \leq b$ and by* (3.4) *in case $a < b \leq a+\lambda$. Then we have*

$$\begin{aligned}&\left|\int_a^b w(t)f(t)\,dt - \int_a^{a+\lambda} f(t)\,dt - T_{w,n}^{[a,b],H^1} + T_{1,n}^{[a,a+\lambda],H^2}\right| \\ &\leq \left\|f^{(n)}\right\|_p \left(\int_a^{\max\{b,a+\lambda\}} |K_n(s)|^q\,ds\right)^{\frac{1}{q}}.\end{aligned} \tag{4.3}$$

The constant $\left(\int_a^{\max\{b,a+\lambda\}} |K_n(s)|^q\,ds\right)^{1/q}$ in the inequality (4.3) *is sharp for $1 < p \leq \infty$ and the best possible for $p=1$.*

Proof. We take $c=a$, $d=a+\lambda$ and $u(t)=1$ in Theorem 6. □

Theorem 8. *Suppose that all assumptions of Theorem 5 hold. Additionally assume (p,q) is a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$, $1/p+1/q=1$. Let $\left|f^{(n)}\right|^p : [a,b]\cup[b-\lambda,b] \to \mathbb{R}$ be an R-integrable function for some $n \geq 1$. Let $K_n(s)$ be defined by* (3.7) *in case $a \leq b-\lambda \leq b$ and by* (3.8) *in case $b-\lambda \leq a \leq b$. Then we have*

$$\begin{aligned}&\left|\int_a^b w(t)f(t)\,dt - \int_{b-\lambda}^b f(t) + T_{1,n}^{[b-\lambda,b],H^2} - T_{w,n}^{[a,b],H^1}\right| \\ &\leq \left\|f^{(n)}\right\|_p \left(\int_{\min\{a,b-\lambda\}}^b |K_n(s)|^q\,ds\right)^{\frac{1}{q}}.\end{aligned} \tag{4.4}$$

The constant $\left(\int_{\min\{a,b-\lambda\}}^{b} |K_n(s)|^q \, ds\right)^{1/q}$ *in the inequality* (4.4) *is sharp for* $1 < p \le \infty$ *and the best possible for* $p = 1$.

Proof. Similar as Theorem 7. □

5. $n-$EXPONETIAL CONVEXITY OF STEFFENSEN'S INEQUALITY BY HERMITE'S POLYNOMIAL

Motivated by inequalities (2.9),(3.2),(3.6), and under assumptions of Theorems 3, 4 and 5, respectively, we define following linear functionals:

$$L_1(f) = \int_a^b w(t) f(t) \, dt - \int_c^d u(t) f(t) \, dt - T_{w,n}^{[a,b],H^1} + T_{u,n}^{[c,d],H^2} \tag{5.1}$$

$$L_2(f) = \int_a^b w(t) f(t) \, dt - \int_a^{a+\lambda} f(t) \, dt - T_{w,n}^{[a,b],H^1} + T_{1,n}^{[a,a+\lambda],H^2} \tag{5.2}$$

$$L_3(f) = \int_{b-\lambda}^b f(t) \, dt - \int_a^b w(t) f(t) \, dt - T_{1,n}^{[b-\lambda,b],H^2} + T_{w,n}^{[a,b],H^1}. \tag{5.3}$$

Also, we define $I_1 = [a,b] \cup [c,d]$, $I_2 = [a,b] \cup [a,a+\lambda]$ and $I_3 = [a,b] \cup [b-\lambda,b]$.

Remark 5.1: Under the assumptions of Theorems 3, 4 and 5 respectively it holds $L_i(f) \ge 0$, $i = 1,2,3$ for all $n-$ convex functions.

First we will state and prove mean value theorems for defined functionals.

Theorem 9. *Let* $f : I_i \to \mathbb{R}$ $(i = 1,2,3)$ *be such that* $f \in C^n(I_i)$. *If inequalities in* (2.8) $(i = 1)$, (3.1) $(i = 2)$ *and* (3.5) $(i = 3)$ *hold, then there exist* $\xi_i \in I_i$ *such that*

$$L_i(f) = f^{(n)}(\xi_i) L_i(\varphi), \quad i = 1,2,3 \tag{5.4}$$

where $\varphi(x) = \frac{x^n}{n!}$.

Proof. Let us denote $m = \min f^{(n)}$ and $M = \max f^{(n)}$. For a given function $f \in C^n(I_i)$ we define functions $F_1, F_2 : I_i \to \mathbb{R}$ with

$$F_1(x) = \frac{Mx^n}{n!} - f(x) \quad \text{and} \quad F_2(x) = f(x) - \frac{mx^n}{n!}.$$

Now $F_1^{(n)}(x) = M - f^{(n)} \ge 0, x \in I_i$, so we conclude $L_i(F_1) \ge 0$ and then $L_i(f) \le M \cdot L_i(\varphi)$. Similarly, from $F_2^{(n)}(x) = f^{(n)}(x) - m \ge 0$ we conclude $m \cdot L_i(\varphi) \le L_i(f)$.

If $L_i(\varphi) = 0$ (5.4) holds for all $\xi_i \in I_i$. Otherwise, $m \le \frac{L_i(f)}{L_i(\varphi)} \le M$. Since $f^{(n)}(x)$ is continuous on I_i there exist $\xi_i \in I_i$ such that (5.4) holds. □

Theorem 10. *Let $f, g : I_i \to \mathbb{R}$ $(i = 1, 2, 3)$ be such that $f, g \in C^n(I_i)$ and $g^{(n)}(x) \neq 0$ for every $x \in I_i$. If inequalities in* (2.8) $(i = 1)$, (3.1) $(i = 2)$ *and* (3.5) $(i = 3)$ *hold, then there exist $\xi_i \in I_i$ such that*

$$\frac{L_i(f)}{L_i(g)} = \frac{f^{(n)}(\xi_i)}{g^{(n)}(\xi_i)}, \quad i = 1, 2, 3. \tag{5.5}$$

Proof. We define functions $\phi_i(x) = f(x)L_i(g) - g(x)L_i(f),\ i = 1, 2, 3$. According to Theorem 9 there exists $\xi_i \in I_i$ such that $L_i(\phi_i) = \phi_i^{(n)}(\xi_i)L_i(\varphi)$. Since $L_i(\phi_i) = 0$ it follows $f^{(n)}(\xi_i)L_i(g) - g^{(n)}(\xi_i)L_i(f) = 0$ and (5.5) is proved. □

Now we are ready to investigate the properties of functional as defined above, regarding $n-$exponential and exponetial convexity. We start this part of the section by giving some definitions and properties which are used frequently in the results (see [6]).

Definition 5.1: A function $\psi : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} \xi_i \xi_j\, \psi\left(\frac{x_i + x_j}{2}\right) \geq 0,$$

hold for all choices $\xi_1, \ldots, \xi_n \in \mathbb{R}$ and all choices $x_1, \ldots, x_n \in I$. A function $\psi : I \to \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 5.2: It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex function in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N},\ k \leq n$.

Definition 5.2: A function $\psi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

A function $\psi : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Remark 5.3: In [4] it is showed that $\psi : I \to \mathbb{R}$ is a log-convex in the Jensen sense if and only if

$$\alpha^2\psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2\psi(y) \geq 0,$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.

A positive function is log-convex if and only if it is 2-exponentially convex.

Proposition 5.1. *If f is a convex function on I and if $x_1 \leq y_1,\ x_2 \leq y_2,\ x_1 \neq x_2,\ y_1 \neq y_2$, then the following inequality is valid*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}.$$

If the function f is concave, the inequality is reversed.

Definition 5.3: Let f be a real-valued function defined on the segment $[a, b]$. The *divided difference* of order n of the function f at distinct points $x_0, ..., x_n \in [a, b]$, is defined recursively (see [2], [8]) by

$$f[x_i] = f(x_i), \ (i = 0, \dots, n)$$

and

$$f[x_0, \dots, x_n] = \frac{f[x_1, \dots, x_n] - f[x_0, \dots, x_{n-1}]}{x_n - x_0}.$$

The value $f[x_0, \dots, x_n]$ is independent of the order of the points $x_0, \dots, x_n$. The definition may be extended to include the case that some (or all) of the points coincide. Assuming that $f^{(j-1)}(x)$ exists, we define

$$f[\underbrace{x, \dots, x}_{j-times}] = \frac{f^{(j-1)}(x)}{(j-1)!}. \tag{5.6}$$

We use an elegant method of producing $n-$ exponentially convex and exponentially convex functions is given in [4]. We use this method to prove the $n-$exponential convexity for above defined functionals. In the sequel the notion log denotes the natural logarithm function.

Theorem 11. *Let $\Omega = \{f_p : p \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I_i, $i = 1, 2, 3$ in $\mathbb{R}$ such that the function $p \mapsto f_p[x_0, \dots, x_m]$ is $n-$exponentially convex in the Jensen sense on J for every $(m+1)$ mutually different points $x_0, \dots, x_m \in I_i$, $i = 1, 2, 3$. Let L_i, $i = 1, 2, 3$ be linear functionals defined by (5.1)-(5.3). Then $p \mapsto L_i(f_p)$ is $n-$exponentially convex function in the Jensen sense on J.*
If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is $n-$exponentially convex on J.

Proof. For $\xi_j \in \mathbb{R}$, $j = 1, \dots, n$ and $p_j \in J$, $j = 1, \dots, n$, we define the function

$$g(x) = \sum_{j,k=1}^{n} \xi_j \xi_k f_{\frac{p_j + p_k}{2}}(x).$$

Using the assumption that the function $p \mapsto f_p[x_0, \dots, x_m]$ is n-exponentially convex in the Jensen sense, we have

$$g[x_0, \dots, x_m] = \sum_{j,k=1}^{n} \xi_j \xi_k f_{\frac{p_j + p_k}{2}}[x_0, \dots, x_m] \geq 0,$$

which in turn implies that g is a m-convex function on J, so it is $L_i(g) \geq 0$, $i = 1, 2, 3$, hence

$$\sum_{j,k=1}^{n} \xi_j \xi_k L_i \left(f_{\frac{p_j + p_k}{2}} \right) \geq 0.$$

We conclude that the function $p \mapsto L_i(f_p)$ is n-exponentially convex on J in the Jensen sense.

If the function $p \mapsto L_i(f_p)$ is also continuous on J, then $p \mapsto L_i(f_p)$ is n-exponentially convex by definition. $\square$

The following corollaries are an immediate consequences of the above theorem:

Corollary 11.1. *Let $\Omega = \{f_p : p \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I_i, $i = 1,2,3$ in $\mathbb{R}$, such that the function $p \mapsto f_p[x_0, \ldots, x_m]$ is exponentially convex in the Jensen sense on J for every $(m+1)$ mutually different points $x_0, \ldots, x_m \in I_i$. Let L_i, $i = 1,2,3$, be linear functionals defined as in* (5.1)-(5.3)*. Then $p \mapsto L_i(f_p)$ is an exponentially convex function in the Jensen sense on J. If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is exponentially convex on J.*

Corollary 11.2. *Let $\Omega = \{f_p : p \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I_i, $i = 1,2,3$ in $\mathbb{R}$, such that the function $p \mapsto f_p[x_0, \ldots, x_m]$ is 2-exponentially convex in the Jensen sense on J for every $(m+1)$ mutually different points $x_0, \ldots, x_m \in I_i$. Let L_i, $i = 1,2,3$ be linear functionals defined as in* (5.1)-(5.3)*. Then the following statements hold:*

(i) If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is 2-exponentially convex function on J. If $p \mapsto L_i(f_p)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true:

$$[L_i(f_s)]^{t-r} \leq [L_i(f_r)]^{t-s} [L_i(f_t)]^{s-r} \tag{5.7}$$

for every choice $r, s, t \in J$, such that $r < s < t$.

(ii) If the function $p \mapsto L_i(f_p)$ is strictly positive and differentiable on J, then for every $p, q, u, v \in J$, such that $p \leq u$ and $q \leq v$, we have

$$\mu_{p,q}(L_i, \Omega) \leq \mu_{u,v}(L_i, \Omega), \tag{5.8}$$

where

$$\mu_{p,q}(L_i, \Omega) = \begin{cases} \left(\frac{L_i(f_p)}{L_i(f_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\frac{d}{dp} L_i(f_p)}{L_i(f_p)}\right), & p = q, \end{cases} \tag{5.9}$$

for $f_p, f_q \in \Omega$.

Proof. (i) This is an immediate consequence of Theorem 11 and Remark 5.3.

(ii) Since $p \mapsto L_i(f_p)$ is positive and continuous, by (i) we have that $p \mapsto L_i(f_p)$ is log-convex on J, that is, the function $p \mapsto \log L_i(f_p)$ is convex on J. Applying Proposition 5.1 we get

$$\frac{\log L_i(f_p) - \log L_i(f_q)}{p - q} \leq \frac{\log L_i(f_u) - \log L_i(f_v)}{u - v}, \tag{5.10}$$

for $p \leq u, q \leq v, p \neq q, u \neq v$. Hence, we conclude that

$$\mu_{p,q}(L_i, \Omega) \leq \mu_{u,v}(L_i, \Omega).$$

Cases $p = q$ and $u = v$ follows from (5.10) as limit cases. $\square$

Remark 5.4: Note that the results from above theorem and corollaries still hold when two of the points $x_0, \ldots, x_m \in I_i,\ i = 1, 2, 3$ coincide, say $x_1 = x_0$, for a family of differentiable functions f_p such that the function $p \mapsto f_p[x_0, \ldots, x_m]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all $m+1$ points coincide for a family of m differentiable functions with the same property. The proofs use (5.6) and suitable characterization of convexity.

6. Applications to Stolarsky type means

In this section we will apply general results obtained in previous section to several families of functions which fulfil conditions of obtained general results. This enable us to construct a concrete examples of exponentially convex functions.

Example 6.1: Consider a family of functions

$$\Omega_1 = \{f_p : \mathbb{R} \to \mathbb{R} : p \in \mathbb{R}\}$$

defined by

$$f_p(x) = \begin{cases} \frac{e^{px}}{p^n}, & p \neq 0, \\ \frac{x^n}{n!}, & p = 0. \end{cases}$$

We have $\frac{d^n f_p}{dx^n}(x) = e^{px} > 0$ which shows that f_p is n-convex on $\mathbb{R}$ for every $p \in \mathbb{R}$ and $p \mapsto \frac{d^n f_p}{dx^n}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 11 we also have that $p \mapsto f_p[x_0, \ldots, x_m]$ is exponentially convex (and so exponentially convex in the Jensen sense). Using Corollary 11.1 we conclude that $p \mapsto L_i(f_p), i = 1, 2, 3$, are exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although mapping $p \mapsto f_p$ is not continuous for $p = 0$), so it is exponentially convex. For this family of functions, $\mu_{p,q}(L_i, \Omega_1),\ i = 1, 2, 3$, from (5.9), becomes

$$\mu_{p,q}(L_i, \Omega_1) = \begin{cases} \left(\frac{L_i(f_p)}{L_i(f_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{L_i(id \cdot f_p)}{L_i(f_p)} - \frac{n}{p}\right), & p = q \neq 0, \\ \exp\left(\frac{1}{n+1} \frac{L_i(id \cdot f_0)}{L_i(f_0)}\right), & p = q = 0, \end{cases}$$

where id is the identity function. Also, by Corollary 11.2 it is monotonic function in parameters p and q.

We observe here that $\left(\frac{\frac{d^n f_p}{dx^n}}{\frac{d^n f_q}{dx^n}}\right)^{\frac{1}{p-q}} (\log x) = x$ so using Theorem 5.5 it follows that:

$$M_{p,q}(L_i, \Omega_1) = \log \mu_{p,q}(L_i, \Omega_1),\ i = 1, 2, 3$$

satisfies

$$\min\{a, b - \lambda, c\} \leq M_{p,q}(L_i, \Omega_1) \leq \max\{b, d, a + \lambda\},\ i = 1, 2, 3.$$

So, $M_{p,q}(L_i, \Omega_1)$ is monotonic mean.

Example 6.2: Consider a family of functions

$$\Omega_2 = \{g_p : (0,\infty) \to \mathbb{R} : p \in \mathbb{R}\}$$

defined by

$$g_p(x) = \begin{cases} \frac{x^p}{p(p-1)\cdots(p-n+1)}, & p \notin \{0,1,\ldots,n-1\}, \\ \frac{x^j \log x}{(-1)^{n-1-j} j!(n-1-j)!}, & p = j \in \{0,1,\ldots,n-1\}. \end{cases}$$

Here, $\frac{d^n g_p}{dx^n}(x) = x^{p-n} > 0$ which shows that g_p is $n-$convex for $x > 0$ and $p \mapsto \frac{d^n g_p}{dx^n}(x)$ is exponentially convex by definition. Arguing as in Example 6.1 we get that the mappings $p \mapsto L_i(g_p), i = 1,2,3$ are exponentially convex. For this family of functions $\mu_{p,q}(L_i, \Omega_2),\ i = 1,2,3$, from (5.9), is now equal to

$$\mu_{p,q}(L_i,\Omega_2) = \begin{cases} \left(\frac{L_i(g_p)}{L_i(g_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left((-1)^{n-1}(n-1)!\frac{L_i(g_0 g_p)}{L_i(g_p)} + \sum\limits_{k=0}^{n-1}\frac{1}{k-p}\right), & \\ & p = q \notin \{0,1,\ldots,n-1\}, \\ \exp\left((-1)^{n-1}(n-1)!\frac{L_i(g_0 g_p)}{2L_i(g_p)} + \sum\limits_{\substack{k=0 \\ k\neq p}}^{n-1}\frac{1}{k-p}\right), & \\ & p = q \in \{0,1,\ldots,n-1\}. \end{cases}$$

Again, using Theorem 10 we conclude that

$$\min\{a, b-\lambda, c\} \leq \left(\frac{L_i(g_p)}{L_i(g_q)}\right)^{\frac{1}{p-q}} \leq \max\{a+\lambda, b, d\},\ i = 1,2,3. \tag{6.1}$$

So, $\mu_{p,q}(L_i,\Omega_2), i = 1,2,3$ is mean.

Example 6.3: Consider a family of functions

$$\Omega_3 = \{\phi_p : (0,\infty) \to \mathbb{R} : p \in (0,\infty)\}$$

defined by

$$\phi_p(x) = \begin{cases} \frac{p^{-x}}{(-\log p)^n}, & p \neq 1 \\ \frac{x^n}{n!}, & p = 1. \end{cases}$$

Since $\frac{d^n \phi_p}{dx^n}(x) = p^{-x}$ is the Laplace transform of a non-negative function (see [9]) it is exponentially convex. Obviously ϕ_p are n-convex functions for every $p > 0$. For this family of functions, $\mu_{p,q}(L_i,\Omega_3), i = 1,2,3$ from (5.9) is equal to

$$\mu_{p,q}(L_i,\Omega_3) = \begin{cases} \left(\frac{L_i(\phi_p)}{L_i(\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(-\frac{L_i(id\cdot\phi_p)}{p\ L_i(\phi_p)} - \frac{n}{p\log p}\right), & p = q \neq 1, \\ \exp\left(-\frac{1}{n+1}\frac{L_i(id\cdot\phi_1)}{L_i(\phi_1)}\right), & p = q = 1, \end{cases}$$

where id is the identity function. This is monotone function in parameters p and q by (5.8). Using Theorem 10 it follows that

$$M_{p,q}(L_i,\Omega_3) = -L(p,q)\log\mu_{p,q}(L_i,\Omega_3),\ i=1,2,3$$

satisfies

$$\min\{a,b-\lambda,c\} \leq M_{p,q}(L_i,\Omega_3) \leq \max\{a+\lambda,b,d\}.$$

So $M_{p,q}(L_i,\Omega_3)$ is monotonic mean. $L(p,q)$ is logarithmic mean defined by

$$L(p,q) = \begin{cases} \frac{p-q}{\log p-\log q}, & p \neq q \\ p, & p=q. \end{cases}$$

Example 6.4: Consider a family of functions

$$\Omega_4 = \{\psi_p : (0,\infty) \to \mathbb{R} : p \in (0,\infty)\}$$

defined by

$$\psi_p(x) = \frac{e^{-x\sqrt{p}}}{(-\sqrt{p})^n}.$$

Since $\frac{d^n\psi_p}{dx^n}(x) = e^{-x\sqrt{p}}$ is the Laplace transform of a non-negative function (see [9]) it is exponentially convex. Obviously ψ_p are n-convex functions for every $p>0$. For this family of functions, $\mu_{p,q}(L_i,\Omega_4), i=1,2,3$ from (5.9) is equal to

$$\mu_{p,q}(L_i,\Omega_4) = \begin{cases} \left(\frac{L_i(\psi_p)}{L_i(\psi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(-\frac{L_i(id\cdot\psi_p)}{2\sqrt{p}L_i(\psi_p)} - \frac{n}{2p}\right), & p=q, \end{cases}$$

where id is the identity function. This is monotone function in parameters p and q by (5.8). Using Theorem 10 it follows that

$$M_{p,q}(L_i,\Omega_4) = -(\sqrt{p}+\sqrt{q})\log\mu_{p,q}(L_i,\Omega_4),\ i=1,2,3$$

satisfies $\min\{a,b-\lambda,c\} \leq M_{p,q}(L_i,\Omega_4) \leq \max\{a+\lambda,b,d\}$, so $M_{p,q}(L_i,\Omega_4)$ is monotonic mean.

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References

[1] R. P. Agarwal, P. J. Y.Wong, *Error Inequalities in Polynomial Interpolation and Their Applications*, Kluwer Academic Publishers, Dordrecht / Boston / London, 1993.

[2] K. E. Atkinson, *An Introduction to Numerical Analysis*, 2nd ed., Wiley, New York, 1989.

[3] P. R. Beesack, *On the Green's function of an N-point boundary value problem*, Pacific J. Math. 12 (1962), 801-812. Kluwer Academic Publishers, Dordrecht / Boston / London, 1993.

[4] J. Jakšetić, J. Pečarić, *Exponential Convexity Method*, J. Convex Anal. (2013), no. 1, 181-197.

[5] A. Yu. Levin, *Some problems bearing on the oscillation of solutions of linear differential equations*, Soviet Math. Dokl., 4(1963), 121-124.

[6] J. Pečarić, J. Perić, *Improvements of the Giaccardi and the Petrović inequality and related results*,An. Univ. Craiova Ser. Mat. Inform., **39**(1) (2012), 65-75.

[7] J.F. Steffensen, *On certain inequalities between mean values and their application to actuarial problems*, Skand. Aktuarietids. (1918), 82–97.

[8] J. E. Pečarić, F. Proschan and Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, Mathematics in science and engineering 187, Academic Press, 1992.
[9] D. V. Widder, *The Laplace transform*, Princeton Univ. Press, New Jersey, 1941.

Faculty of Mechanical Engineering and Naval Architecture,
University of Zagreb, Ivana Lučića 5, 10000 Zagreb, Croatia
E-mail address: julije@math.hr

Faculty of Textile Technology, University of Zagreb,
Prilaz baruna Filipovića 28a, 10000 Zagreb, Croatia
E-mail address: pecaric@element.hr

Faculty of Civil Engineering, University of Rijeka,
Radmile Matejčić 3, 51000 Rijeka, Croatia
E-mail address: anamarija.perusic@gradri.hr