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ON SYMMETRIC (71,35,17) DESIGNS

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ABSTRACT. We have enumerated all Hadamard (71,35,17) designs having a nonabelian automorphism group of order 34 acting in such a way that an involution fixes exactly seven points. We have also determined the full automorphism groups of constructed designs and their derived and residual designs.

1. INTRODUCTION AND PRELIMINARIES

A 2-(v,k,λ) design is a finite incidence structure $(\mathcal{P},\mathcal{B},I)$, where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and $I \subseteq \mathcal{P} \times \mathcal{B}$, with the following properties:

1.: $|\mathcal{P}| = v$,
2.: every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$,
3.: every pair of distinct elements of $\mathcal{P}$ is incident with exactly λ elements of $\mathcal{B}$.

The elements of the set $\mathcal{P}$ are called points and the elements of the set $\mathcal{B}$ are called blocks. Such designs are often called block designs.

Given two designs $\mathcal{D}_1 = (\mathcal{P}_1,\mathcal{B}_1,I_1)$ and $\mathcal{D}_2 = (\mathcal{P}_2,\mathcal{B}_2,I_2)$, an isomorphism from $\mathcal{D}_1$ onto $\mathcal{D}_2$ is a bijection which maps points onto points and blocks onto blocks preserving the incidence relation. An isomorphism from a design $\mathcal{D}$ onto itself is called an automorphism of $\mathcal{D}$. The set of all automorphisms of the design $\mathcal{D}$ forms a group; it is called the full automorphism group of $\mathcal{D}$ and denoted by $Aut\mathcal{D}$. A symmetric (v,k,λ) design is a 2-(v,k,λ) design with $|\mathcal{P}| = |\mathcal{B}|$.

Let $\mathcal{D} = (\mathcal{P},\mathcal{B},I)$ be a symmetric (v,k,λ) design and $G \leq Aut\mathcal{D}$. The group action of G produces the same number of point and block orbits (see [9, Theorem 3.3]). We denote that number by t, the point orbits by $\mathcal{P}_1,\ldots,\mathcal{P}_t$, the block orbits by $\mathcal{B}_1,\ldots,\mathcal{B}_t$, and put $|\mathcal{P}_r| = \omega_r$ and $|\mathcal{B}_i| = \Omega_i$. We shall denote the points of the orbit $\mathcal{P}_r$ by $r_0,\ldots,r_{\omega_r-1}$, (i.e. $\mathcal{P}_r = \{r_0,\ldots,r_{\omega_r-1}\}$). Further, we denote by γ_{ir} the number of points of $\mathcal{P}_r$ which are incident with a representative of the block

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orbit $\mathcal{B}_i$. For those numbers the following equalities hold (see [5]):

$$\sum_{r=1}^{t} \gamma_{ir} = k\,, \tag{1}$$

$$\sum_{r=1}^{t} \frac{\Omega_j}{\omega_r} \gamma_{ir}\gamma_{jr} = \lambda\Omega_j + \delta_{ij} \cdot (k-\lambda)\,. \tag{2}$$

Definition 1. *A $(t \times t)$-matrix (γ_{ir}) with entries satisfying conditions (1) and (2) is called an orbit structure for the parameters (v, k, λ) and orbit lengths distributions $(\omega_1, \ldots, \omega_t)$, $(\Omega_1, \ldots, \Omega_t)$.*

The algorithm used for constructing symmetric (v, k, λ) designs $(\mathcal{P}, \mathcal{B}, I)$ admitting presumed automorphism group G is described in details in [5]. More about the elimination of isomorphic structures using elements of the normalizer of the group G one can find in [4]. Here we give a short explanation of the algorithm:

ALGORITHM

The first step - when constructing designs for given parameters and orbit lengths distributions - is to find all compatible orbit structures (γ_{ir}). Mutually isomorphic orbit structures would lead to mutually isomorphic symmetric designs. Therefore, during the construction of orbit structures we use elements of the normalizer of the group G in the group $S = S(\mathcal{P}) \times S(\mathcal{B})$ for elimination of isomorphic orbit structures, taking from each $N_S(G)$-orbit of orbit structures the representative which is first in the reverse lexicographical order.

The next step, called indexing, consists in determining exactly which points from the point orbit $\mathcal{P}_r$ are incident with a fixed representative of the block orbit $\mathcal{B}_i$ for each number γ_{ir}. In this step we also use the elements of the group $N_S(G)$ for elimination of mutually isomorphic structures, applying permutations from $N_S(G)$ which induce automorphisms of the orbit structure (γ_{ir}). At the end of this step, all symmetric designs with given parameters admitting an automorphism group G acting with presumed orbit lengths distributions will be constructed.

Symmetric designs described in this article are obtained by using the computer programs written in programming language **C** which we have developed following the above algorithm.

Definition 2. *The set of those indices of points of the orbit $\mathcal{P}_r$ which are incident with a fixed representative of the block orbit $\mathcal{B}_i$ is called the index set for the position (i, r) of the orbit structure and the given representative.*

A Hadamard matrix of order m is an $(m \times m)$-matrix $H = (h_{i,j})$, $h_{i,j} \in \{-1, 1\}$, satisfying $HH^T = H^TH = mI$, where I is the unit matrix. From each Hadamard matrix of order m one can obtain a symmetric $(m-1, \frac{1}{2}m-1, \frac{1}{4}m-1)$ design (see

[9]). Also, from any symmetric $(m-1, \frac{1}{2}m-1, \frac{1}{4}m-1)$ design we can recover a Hadamard matrix. Symmetric designs with parameters $(m-1, \frac{1}{2}m-1, \frac{1}{4}m-1)$ are called Hadamard designs.

It is known that Hadamard matrices of order 72 could be obtained from Hadamard matrices of order 36, using Kronecker product. As far as we know, symmetric $(71, 35, 17)$ designs constructed from such Hadamard matrices of order 36 have not been investigated and classified. According to [10] only one symmetric $(71, 35, 17)$ design has been known until 1996, and that had been obtained via cyclic difference set (see [7]). Further 11 mutually nonisomorphic symmetric $(71, 35, 17)$ designs, admitting an automorphism group isomorphic to $Frob_{7\cdot 3}$ or $Frob_{17\cdot 8}$, have been constructed lately (see [1], [2]).

The aim of this article is to construct all symmetric $(71, 35, 17)$ designs having a nonabelian automorphism group of order 34, acting in such a way that a permutation of order 2 fixes exactly 7 points. We have also determined the full automorphism groups of these designs and their derived and residual designs.

For further basic definitions and construction procedures we refer the reader to [6] and [14].

2. A Nonabelian Group of Order 34 Acting on Symmetric (71,35,17) Designs

Let $\mathcal{D}$ be a symmetric (71,35,17) design and G a Frobenius group of order 34, further denoted by $Frob_{17\cdot 2}$. Since there is only one isomorphism class of nonabelian groups of order 34 we may write

$$G = \langle \rho, \sigma \mid \rho^{17} = 1, \sigma^2 = 1, \rho^{\sigma} = \rho^{16} \rangle.$$

Lemma 1. *Let $\mathcal{D}$ be a symmetric $(71, 35, 17)$ design and let $\langle \rho \rangle$ be a subgroup of $Aut\mathcal{D}$. If $|\langle \rho \rangle| = 17$, then $\langle \rho \rangle$ fixes precisely three points and three blocks of $\mathcal{D}$.*

Proof. By [9, Theorem 3.1], the group $\langle \rho \rangle$ fixes the same number of points and blocks. Denote that number by f. Obviously, $f \equiv 71 \,(mod\, 17)$, i.e., $f \equiv 3 \,(mod\, 17)$. Using the formula $f \leq k + \sqrt{k - \lambda}$ of [9, Corollary 3.7], we get $f \in \{3, 20, 37\}$. For $f \in \{20, 37\}$ one cannot solve the equations (1) and (2). For $f = 3$, solving equations (1) and (2) by the method described in [5], one can get up to isomorphism and duality exactly two orbit structures:

OS1	1	1	1	17	17	17	17
1	1	0	0	17	17	0	0
1	0	1	0	17	0	17	0
1	0	0	1	17	0	0	17
17	1	1	1	8	8	8	8
17	1	0	0	8	8	9	9
17	0	1	0	8	9	8	9
17	0	0	1	8	9	9	8

OS2	1	1	1	17	17	17	17
1	1	0	0	17	17	0	0
1	0	1	0	17	0	17	0
1	0	0	1	17	0	0	17
17	1	1	0	8	8	8	9
17	1	0	1	8	8	9	8
17	0	1	1	8	9	8	8
17	0	0	0	8	9	9	9

The orbit structure OS1 is self-dual, and the structure OS2 is non-self-dual. □

Let G be an automorphism group of a symmetric design $\mathcal{D}$, acting with orbit lengths distributions $(\omega_1, \dots, \omega_t)$, $(\Omega_1, \dots, \Omega_t)$. Automorphism group G is said to be semistandard if, after possibly renumbering orbits, we have $\omega_i = \Omega_i$, for $i = 1, \dots, t$.

Let $G = \langle \rho, \sigma \rangle$ be the Frobenius group of order 34 defined above, $\mathcal{D}$ a symmetric $(71, 35, 17)$ design, and $G \leq Aut\mathcal{D}$ acting in such a way that the permutation σ fixes exactly 7 points. The Frobenius kernel $\langle \rho \rangle$ of order 17 acts on $\mathcal{D}$ with orbit lengths distribution $(1, 1, 1, 17, 17, 17, 17)$. Since $\langle \rho \rangle \lhd G$, the element σ of order 2 maps $\langle \rho \rangle$-orbits onto $\langle \rho \rangle$-orbits. Therefore, the group G also acts on the design $\mathcal{D}$ semistandardly with orbit lengths distribution $(1, 1, 1, 17, 17, 17, 17)$.

Theorem 1. *Up to isomorphism there are precisely forty-five symmetric designs with parameters* $(71, 35, 17)$ *admitting a nonabelian automorphism group of order* 34, *acting in such a way that the involution fixes exactly* 7 *points. Three of these designs have* $Frob_{17\cdot 8}$ *as the full automorphism group, and six designs have the full automorphism group isomorphic to the group* $Frob_{17\cdot 4}$. *The full automorphism groups of the other thirty-six designs are isomorphic to the group* $Frob_{17\cdot 2}$.

Proof. We denote the points by $1_0, 2_0, 3_0, 4_i, 5_i, 6_i, 7_i$, $i = 0, 1, \dots, 16$ and put $G = \langle \rho, \sigma \rangle$, where the generators for G are permutations defined as follows:

$$\rho = (1_0)(2_0)(3_0)(I_0, \dots, I_{16}), \; I = 4, 5, 6, 7,$$
$$\sigma = (1_0)(2_0)(3_0)(K_0)(K_1, K_{16})(K_2, K_{15})(K_3, K_{14})(K_4, K_{13})(K_5, K_{12})$$
$$(K_6, K_{11})(K_7, K_{10})(K_8, K_9), \; K = 4, 5, 6, 7.$$

Indexing the fixed part of an orbit structure is a trivial task. Therefore, we shall consider the right-lower part of the orbit structures of order 4. To eliminate isomorphic structures during the indexing process we have used the permutations α_i, $i = 2, 3, 4, 5, 6$, which - on each $\langle \rho \rangle$-orbit of points of length 17 - acts as $\alpha_i(x) = i \cdot x \; (mod \; 17)$, and - in addition - automorphisms of the orbit structures OS1 and OS2.

As representatives for the block orbits of length 17 we chose blocks fixed by $\langle\sigma\rangle$. Therefore, the index sets must be unions of the sets $\{0\}$, $\{1,16\}$, $\{2,15\}$, $\{3,14\}$, $\{4,13\}$, $\{5,12\}$, $\{6,11\}$, $\{7,10\}$ and $\{8,9\}$. The index sets which could occur in the designs constructed from OS1 and OS2 are among the following:

$$0 = \{0,1,2,3,4,13,14,15,16\},\ 1 = \{1,2,3,5,12,14,15,16\}, \dots ,$$
$$69 = \{5,6,7,8,9,10,11,12\},\ 70 = \{0,1,2,3,4,134,14,15,16\},$$
$$71 = \{0,1,2,3,5,12,14,15,16\}, \dots ,\ 139 = \{0,5,6,7,8,9,10,11,12\}.$$

The indexing process of the orbit structure OS1 led to fifteen designs, denoted by $\mathcal{D}_1, \dots, \mathcal{D}_{15}$. These fifteen designs have fifteen different statistics of intesection of any four blocks. So, the designs are pairwise nonisomorphic.

Using a computer program by V. Krčadinac (see [8] and [11]) we get that the pairs of mutually dual designs are $(\mathcal{D}_1, \mathcal{D}_2)$, $(\mathcal{D}_3, \mathcal{D}_4)$, $(\mathcal{D}_5, \mathcal{D}_6)$, $(\mathcal{D}_7, \mathcal{D}_8)$, $(\mathcal{D}_9, \mathcal{D}_{10})$ and $(\mathcal{D}_{14}, \mathcal{D}_{15})$. The designs $\mathcal{D}_{11}$, $\mathcal{D}_{12}$ and $\mathcal{D}_{13}$ are self-dual. A computer program by Vladimir D. Tonchev [13] computes the orders as well as the generators of the full automorphism groups of these designs. The full automorphism group of the design $\mathcal{D}_{12}$ is isomorphic to the group $Frob_{17\cdot 8}$ of order 136, and the full automorphism groups of $\mathcal{D}_{14}$ and $\mathcal{D}_{15}$ are isomorphic to the group $Frob_{17\cdot 4}$. The other designs have the full automorphism groups isomorphic to the group $Frob_{17\cdot 2}$. We present the symmetric designs $\mathcal{D}_1$, $\mathcal{D}_{12}$ and $\mathcal{D}_{14}$ by (4×4)-matrices of index sets as follows:

$$\mathcal{D}_1 \qquad \begin{pmatrix} 2 & 32 & 50 & 56 \\ 50 & 56 & 137 & 107 \\ 56 & 89 & 32 & 137 \\ 32 & 137 & 83 & 50 \end{pmatrix}$$

$$\mathcal{D}_{12} \qquad \begin{pmatrix} 8 & 8 & 61 & 61 \\ 61 & 61 & 131 & 131 \\ 8 & 131 & 61 & 78 \\ 61 & 78 & 131 & 8 \end{pmatrix}$$

$$\mathcal{D}_{14} \qquad \begin{pmatrix} 15 & 28 & 41 & 54 \\ 41 & 54 & 124 & 111 \\ 54 & 98 & 28 & 124 \\ 28 & 124 & 85 & 41 \end{pmatrix}$$

From these matrices it is easy to obtain incidence matrices of designs. The index sets wich occur in the designs $\mathcal{D}_1$, $\mathcal{D}_{12}$ and $\mathcal{D}_{14}$ are:

$$2 = \{1,2,3,6,11,14,15,16\},\ 8 = \{1,2,4,8,9,13,15,16\},$$
$$15 = \{1,3,4,5,12,13,14,16\},\ 28 = \{1,4,6,7,10,11,13,16\},$$
$$32 = \{1,5,6,8,9,11,12,16\},\ 41 = \{2,3,5,8,9,12,14,15\},$$
$$50 = \{2,4,7,8,9,10,13,15\},\ 54 = \{2,6,7,8,9,10,11,15\},$$
$$56 = \{3,4,5,7,10,12,13,14\},\ 61 = \{3,5,6,7,10,11,12,14\},$$
$$78 = \{0,1,2,4,8,9,13,15,16\},\ 83 = \{0,1,2,6,8,9,11,15,16\},$$
$$85 = \{0,1,3,4,5,12,13,14,16\},\ 89 = \{0,1,3,5,6,11,12,14,16\},$$
$$98 = \{0,1,4,6,7,10,11,13,16\},\ 107 = \{0,2,3,4,7,10,13,14,15\},$$
$$111 = \{0,2,3,5,8,9,12,14,15\},\ 124 = \{0,2,6,7,8,9,10,11,15\},$$
$$131 = \{0,3,5,6,7,10,11,12,14\},\ 137 = \{0,4,5,7,8,9,10,12,13\}.$$

The orbit structure OS2 also leads to fifteen designs, denoted by $\mathcal{D}_{15},\dots,\mathcal{D}_{30}$. The full automorphism group of the design $\mathcal{D}_{28}$ is isomorphic to the group $Frob_{17\cdot 8}$ of order 136, and the full automorphism groups of $\mathcal{D}_{29}$ and $\mathcal{D}_{30}$ are isomorphic to the group $Frob_{17\cdot 4}$. The other 12 designs have the full automorphism groups isomorphic to the group $Frob_{17\cdot 2}$.

The orbit structure OS2 is not self-dual. Therefore, the dual structure of OS2 must produce also fifteen designs, which are dual to the designs obtained from the structure OS2. These designs are denoted by $\mathcal{D}_{31},\dots,\mathcal{D}_{45}$. The forty-five constructed symmetric designs have forty-five different statistics of intersection of any four blocks. □

Remark 1. Application of the computer program by Tonchev [13] yields that the 2-ranks of the forty-five constructed symmetric designs are equal to 36.

Remark 2. Since the designs having the group $Frob_{17\cdot 8}$ as the full automorphism group are isomorphic to the designs described in [1], Theorem 1 implies that there are at least 54 symmetric (71,35,17) designs.

3. On 2-(35,17,16) and 2-(36,18,17) Designs

Excluding from a symmetric (71,35,17) design $\mathcal{D}$ the block x and all the points that do not belong to that block, one obtains its derived 2-(35,17,16) design $\mathcal{D}_x$. Also, excluding the block x and all the points belonging to that block from the design $\mathcal{D}$, one obtains its residual 2-(36,18,17) design $\mathcal{D}^x$ (see [9]).

According to [10], there are at least 1854 block designs with parameters (35,17,16), and at least 91 block designs with parameters (36,18,17).

Our aim is to investigate the full automorphism groups of all pairwise non-isomorphic derived and residual designs obtained from the symmetric designs $\mathcal{D}_1,\dots,\mathcal{D}_{45}$. Because of the following corollary, it suffice to consider the derived and the residual designs with respect to the block orbits representatives of the full automorphism groups of the constructed (71,35,17) designs.

Corollary 1. Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a symmetric design, $x, x' \in \mathcal{B}$ and $G \leq Aut\mathcal{D}$. If $x' \in xG$, then $\mathcal{D}_x \cong \mathcal{D}_{x'}$ and $\mathcal{D}^x \cong \mathcal{D}^{x'}$.

Proof. [3],[Corollary 1]. □

We have obtained 230 pairwise nonisomorphic derived designs. Further investigation of the constructed 2-(35,17,16) designs led us to the following results:

The order of the full automorphism group	*The full automorphism group structure*	*Number of designs*
272	$Frob_{17 \cdot 16}$	4
272	$Frob_{17 \cdot 8} \times Z_2$	1
136	$Frob_{17 \cdot 8}$	5
68	$Frob_{17 \cdot 4}$	4
34	$Frob_{17 \cdot 2}$	36
8	Z_8	12
4	Z_4	24
2	Z_2	144

Among the constructed 2-(35,17,16) designs there is one quasi-symmetric design, obtained from the design $\mathcal{D}_{12}$ excluding the second fixed block. The full automorphism group of that quasi-symmetric design is isomorphic to the group $Frob_{17 \cdot 8}$.

Investigating all residual designs with respect to the block orbits representatives, we have constructed all pairwise nonisomorphic 2-(36,18,17) designs that can be obtained from the constructed symmetric (71,35,17) designs $\mathcal{D}_1, \ldots, \mathcal{D}_{45}$. We have obtained 270 pairwise nonisomorphic 2-(36,18,17) designs. Further investigation of the constructed 2-(36,18,17) designs led us to the following results:

The order of the full automorphism group	*The full automorphism group structure*	*Number of designs*
272	$Frob_{17 \cdot 16}$	1
272	$Frob_{17 \cdot 8} \times Z_2$	1
136	$Frob_{17 \cdot 8}$	4
68	$Frob_{17 \cdot 4}$	12
34	$Frob_{17 \cdot 2}$	72
8	Z_8	12
4	Z_4	24
2	Z_2	144

It is clear that among the constructed 2-(36,18,17) designs there is one quasi-symmetric designs, obtained from the design $\mathcal{D}_{12}$ excluding the second fixed block. The full automorphism group of that quasi-symmetric residual design is isomorphic to the group $Frob_{17\cdot 16}$.

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