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Several Inequalities for Operator Monotone Functions on Finite Intervals

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Abstract: In this paper we show that, if the function $f : (-1,1) \rightarrow \mathbb{R}$ is operator monotone in $(-1,1)$, then there exists a positive measure μ on $[-1,1]$ such that $[f(B) - f(A)](B - A) = f'(0) \int_{-1}^1 \left(\int_0^1 \left[(1 - \lambda((1-t)A + tB))^{-1} (B - A) \right]^2 dt \right) d\mu(\lambda)$ for all A, B with $S_p(A), S_p(B) \subset (-1,1)$. Some necessary and sufficient conditions for the operators A, B with $\text{Sp}(A), \text{Sp}(B) \subset (-1,1)$ such that the inequality $f(B)B + f(A)A \geq f(A)B + f(B)A$ holds for any operator monotone function f on $(-1,1)$ are also given.

Key words and phrases: Operator monotone functions, Integral inequalities, Operator inequality.

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