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INEQUALITIES FOR HYPO-q-NORMS ON A CARTESIAN PRODUCT OF INNER PRODUCT SPACES

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Abstract. In this paper we introduce the hypo-q-norms on a Cartesian product of inner product spaces. A representation of these norms in terms of inner products, the equivalence with the q-norms on a Cartesian product and some reverse inequalities obtained via the scalar Shisha-Mond, Birnacki et al., Grüss type inequalities, Boas-Bellman and Bombieri type inequalities are also given.

1. Introduction

Let $(E, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$. On $\mathbb{K}^n$ endowed with the canonical linear structure we consider a norm $\|\cdot\|_n$ and the unit ball

$$\mathbb{B}\left(\|\cdot\|_n\right) := \left\{\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^n | \|\lambda\|_n \leq 1\right\}.$$

As an example of such norms we should mention the usual *p-norms*

$$\|\boldsymbol{\lambda}\|_{n,p} := \begin{cases} \max\left\{|\lambda_1|, \ldots, |\lambda_n|\right\} & \text{if } p = \infty; \\ \\ \left(\sum_{k=1}^n |\lambda_k|^p\right)^{\frac{1}{p}} & \text{if } p \in [1, \infty). \end{cases} \tag{1.1}$$

The *Euclidean norm* is obtained for $p = 2$, i.e.,

$$\|\boldsymbol{\lambda}\|_{n,2} = \left(\sum_{k=1}^n |\lambda_k|^2\right)^{\frac{1}{2}}.$$

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It is well known that on $E^n := E \times \cdots \times E$ endowed with the canonical linear structure we can define the following *p-norms*:

$$\|\mathbf{x}\|_{n,p} := \begin{cases} \max\{\|x_1\|, \ldots, \|x_n\|\} & \text{if } \ p = \infty; \\ \\ \left(\sum_{k=1}^{n} \|x_k\|^p\right)^{\frac{1}{p}} & \text{if } \ p \in [1, \infty); \end{cases} \tag{1.2}$$

where $\mathbf{x} = (x_1, \ldots, x_n) \in E^n$.

Following [9], for a given norm $\|\cdot\|_n$ on $\mathbb{K}^n$, we define the functional $\|\cdot\|_{h,n} : E^n \to [0, \infty)$ given by

$$\|\mathbf{x}\|_{h,n} := \sup_{\lambda \in B\left(\|\cdot\|_n\right)} \left\| \sum_{j=1}^{n} \lambda_j x_j \right\|, \tag{1.3}$$

where $\mathbf{x} = (x_1, \ldots, x_n) \in E^n$.

It is easy to see, by the properties of the norm $\|\cdot\|$, that:

(i) $\|\mathbf{x}\|_{h,n} \geq 0$ for any $\mathbf{x} \in E^n$;
(ii) $\|\mathbf{x} + \mathbf{y}\|_{h,n} \leq \|\mathbf{x}\|_{h,n} + \|\mathbf{y}\|_{h,n}$ for any $\mathbf{x}$, $\mathbf{y} \in E^n$;
(iii) $\|\alpha \mathbf{x}\|_{h,n} = |\alpha| \|\mathbf{x}\|_{h,n}$ for each $\alpha \in \mathbb{K}$ and $\mathbf{x} \in E^n$;

and therefore $\|\cdot\|_{h,n}$ is a *semi-norm* on E^n. This will be called the *hypo-semi-norm* generated by the norm $\|\cdot\|_n$ on E^n.

We observe that $\|\mathbf{x}\|_{h,n} = 0$ if and only if $\sum_{j=1}^{n} \lambda_j x_j = 0$ for any $(\lambda_1, \ldots, \lambda_n) \in B(\|\cdot\|_n)$. If there exists $\lambda_1^0, \ldots, \lambda_n^0 \neq 0$ such that $\left(\lambda_1^0, 0, \ldots, 0\right)$, $\left(0, \lambda_2^0, \ldots, 0\right), \ldots, \left(0, 0, \ldots, \lambda_n^0\right) \in B(\|\cdot\|_n)$ then the semi-norm generated by $\|\cdot\|_n$ is a *norm* on E^n.

If by $\mathbb{B}_{n,p}$ with $p \in [1, \infty]$ we denote the balls generated by the p-norms $\|\cdot\|_{n,p}$ on $\mathbb{K}^n$, then we can obtain the following *hypo-q-norms* on E^n :

$$\|\mathbf{x}\|_{h,n,q} := \sup_{\breve{}\in \mathbb{B}_{n,p}} \left\| \sum_{j=1}^{n} \lambda_j x_j \right\|, \tag{1.4}$$

with $q > 1$ and $\frac{1}{q} + \frac{1}{p} = 1$ if $p > 1$, $q = 1$ if $p = \infty$ and $q = \infty$ if $p = 1$.

For $p = 2$, we have the Euclidean ball in $\mathbb{K}^n$, which we denote by $\mathbb{B}_n$, $\mathbb{B}_n = \left\{ \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^n \left| \sum_{i=1}^{n} |\lambda_i|^2 \leq 1 \right. \right\}$ that generates the *hypo-Euclidean norm* on E^n, i.e.,

$$\|\mathbf{x}\|_{h,e} := \sup_{\breve{}\in \mathbb{B}_n} \left\| \sum_{j=1}^{n} \lambda_j x_j \right\|. \tag{1.5}$$

Moreover, if $E = H$, H is a inner product space over $\mathbb{K}$, then the *hypo-Euclidean norm* on H^n will be denoted simply by

$$\|\mathbf{x}\|_e := \sup_{\breve{\ } \in \mathbb{B}_n} \left\| \sum_{j=1}^{n} \lambda_j x_j \right\|. \tag{1.6}$$

Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{K}$ and $n \in \mathbb{N}$, $n \geq 1$. In the Cartesian product $H^n := H \times \cdots \times H$, for the $n-$tuples of vectors $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in H^n$, we can define the inner product $\langle \cdot, \cdot \rangle$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{j=1}^{n} \langle x_j, y_j \rangle, \qquad \mathbf{x},\ \mathbf{y} \in H^n, \tag{1.7}$$

which generates the Euclidean norm $\|\cdot\|_2$ on H^n, i.e.,

$$\|\mathbf{x}\|_2 := \left(\sum_{j=1}^{n} \|x_j\|^2 \right)^{\frac{1}{2}}, \qquad \mathbf{x} \in H^n. \tag{1.8}$$

The following result established in [9] connects the usual Euclidean norm $\|\cdot\|$ with the hypo-Euclidean norm $\|\cdot\|_e$.

Theorem 1 (Dragomir, 2007, [9])**.** *For any $\mathbf{x} \in H^n$ we have the inequalities*

$$\frac{1}{\sqrt{n}} \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_e \leq \|\mathbf{x}\|_2, \tag{1.9}$$

i.e., $\|\cdot\|_2$ and $\|\cdot\|_e$ are equivalent norms on H^n.

The following representation result for the hypo-Euclidean norm plays a key role in obtaining various bounds for this norm:

Theorem 2 (Dragomir, 2007, [9])**.** *For any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$, we have*

$$\|\mathbf{x}\|_e = \sup_{\|y\| \leq 1} \left(\sum_{j=1}^{n} |\langle x_j, y \rangle|^2 \right)^{\frac{1}{2}}. \tag{1.10}$$

Motivated by the above results, in this paper we introduce the hypo-q-norms on a Cartesian product of inner product spaces. A representation of these norms in terms of inner products, the equivalence with the q-norms on a Cartesian product and some reverse inequalities obtained via the scalar Shisha-Mond, Birnacki et al. and other Grüss type inequalities are also given.

2. General Results

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. We have the following representation result for the *hypo-q-norms* on H^n.

Theorem 3. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. For any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$, we have*

$$\|\mathbf{x}\|_{h,n,q} = \sup_{\|y\| \le 1} \left\{ \left(\sum_{j=1}^{n} |\langle x_j, y \rangle|^q \right)^{1/q} \right\} \tag{2.1}$$

where $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|\mathbf{x}\|_{h,n,1} = \sup_{\|y\| \le 1} \left\{ \sum_{j=1}^{n} |\langle x_j, y \rangle| \right\} \tag{2.2}$$

and

$$\|\mathbf{x}\|_{h,n,\infty} = \|\mathbf{x}\|_{n,\infty} = \max_{j \in \{1,\ldots,n\}} \{\|x_j\|\}. \tag{2.3}$$

In particular,

$$\|\mathbf{x}\|_{h,e} = \sup_{\|y\| \le 1} \left\{ \left(\sum_{j=1}^{n} |\langle x_j, y \rangle|^2 \right)^{1/2} \right\}. \tag{2.4}$$

Proof. Using Hölder's discrete inequality for $p,\ q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\left| \sum_{j=1}^{n} \alpha_j \beta_j \right| \le \left(\sum_{j=1}^{n} |\alpha_j|^p \right)^{1/p} \left(\sum_{j=1}^{n} |\beta_j|^q \right)^{1/q},$$

which implies that

$$\sup_{\|\alpha\|_p \le 1} \left| \sum_{j=1}^{n} \alpha_j \beta_j \right| \le \|\beta\|_q \tag{2.5}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$.

For $(\beta_1, \ldots, \beta_n) \ne 0$, consider $\alpha = (\alpha_1, \ldots, \alpha_n)$ with

$$\alpha_j := \frac{\overline{\beta_j} \, |\beta_j|^{q-2}}{\left(\sum_{k=1}^{n} |\beta_k|^q\right)^{1/p}}$$

for those j for which $\beta_j \ne 0$ and $\alpha_j = 0$, for the rest.

We observe that

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| = \left|\sum_{j=1}^{n}\frac{\overline{\beta_j}\,|\beta_j|^{q-2}}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)^{1/p}}\beta_j\right| = \frac{\sum_{j=1}^{n}|\beta_j|^q}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)^{1/p}}$$
$$= \left(\sum_{j=1}^{n}|\beta_j|^q\right)^{1/q} = \|\beta\|_q$$

and

$$\|\alpha\|_p^p = \sum_{j=1}^{n}|\alpha_j|^p = \sum_{j=1}^{n}\frac{\left|\overline{\beta_j}\,|\beta_j|^{q-2}\right|^p}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)} = \sum_{j=1}^{n}\frac{\left(|\beta_j|^{q-1}\right)^p}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)}$$
$$= \sum_{j=1}^{n}\frac{|\beta_j|^{qp-p}}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)} = \sum_{j=1}^{n}\frac{|\beta_j|^q}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)} = 1.$$

Therefore, by (2.5) we have the representation

$$\sup_{\|\alpha\|_p\le 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| = \|\beta\|_q \tag{2.6}$$

for any $\beta = (\beta_1,\ldots,\beta_n)\in\mathbb{K}^n$.

By the properties of inner product, we have for any $u\in H$, $u\neq 0$ that

$$\|u\| = \sup_{\|y\|\le 1}|\langle u,y\rangle|\,. \tag{2.7}$$

Let $\alpha = (\alpha_1,\ldots,\alpha_n)\in\mathbb{K}^n$ and $\mathbf{x}\in H^n$ with $\mathbf{x} = (x_1,\ldots,x_n)$. Then by (2.7) we have

$$\left\|\sum_{j=1}^{n}\alpha_j x_j\right\| = \sup_{\|y\|\le 1}\left|\left\langle\sum_{j=1}^{n}\alpha_j x_j, y\right\rangle\right| = \sup_{\|y\|\le 1}\left|\sum_{j=1}^{n}\alpha_j\langle x_j,y\rangle\right|. \tag{2.8}$$

By taking the supremum in this equality we have

$$\sup_{\|\alpha\|_p\le 1}\left\|\sum_{j=1}^{n}\alpha_j x_j\right\| = \sup_{\|\alpha\|_p\le 1}\left(\sup_{\|y\|\le 1}\left|\sum_{j=1}^{n}\alpha_j\langle x_j,y\rangle\right|\right)$$
$$= \sup_{\|y\|\le 1}\left(\sup_{\|\alpha\|_p\le 1}\left|\sum_{j=1}^{n}\alpha_j\langle x_j,y\rangle\right|\right) = \sup_{\|y\|\le 1}\left(\sum_{j=1}^{n}|\langle x_j,y\rangle|^q\right)^{1/2},$$

where for the last equality we used the representation (2.6).

This proves (2.1).

Using the properties of the modulus, we have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|\leq\max_{j\in\{1,\dots,n\}}|\alpha_j|\sum_{j=1}^{n}|\beta_j|,$$

which implies that

$$\sup_{\|\alpha\|_\infty\leq 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|\leq\|\beta\|_1, \tag{2.9}$$

where $\alpha=(\alpha_1,\dots,\alpha_n)$ and $\beta=(\beta_1,\dots,\beta_n)$.

For $(\beta_1,\dots,\beta_n)\neq 0$, consider $\alpha=(\alpha_1,\dots,\alpha_n)$ with $\alpha_j:=\frac{\overline{\beta_j}}{|\beta_j|}$ for those j for which $\beta_j\neq 0$ and $\alpha_j=0$, for the rest.

We have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|=\left|\sum_{j=1}^{n}\frac{\overline{\beta_j}}{|\beta_j|}\beta_j\right|=\sum_{j=1}^{n}|\beta_j|=\|\beta\|_1$$

and

$$\|\alpha\|_\infty=\max_{j\in\{1,\dots,n\}}|\alpha_j|=\max_{j\in\{1,\dots,n\}}\left|\frac{\overline{\beta_j}}{|\beta_j|}\right|=1$$

and by (2.9) we get the representation

$$\sup_{\|\alpha\|_\infty\leq 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|=\|\beta\|_1 \tag{2.10}$$

for any $\beta=(\beta_1,\dots,\beta_n)\in\mathbb{K}^n$.

By taking the supremum in the equality (2.8) we have

$$\begin{aligned}\sup_{\|\alpha\|_\infty\leq 1}\left\|\sum_{j=1}^{n}\alpha_j x_j\right\| &= \sup_{\|\alpha\|_\infty\leq 1}\left(\sup_{\|y\|\leq 1}\left|\sum_{j=1}^{n}\alpha_j\langle x_j,y\rangle\right|\right)\\ &= \sup_{\|y\|\leq 1}\left(\sup_{\|\alpha\|_\infty\leq 1}\left|\sum_{j=1}^{n}\alpha_j\langle x_j,y\rangle\right|\right)=\sup_{\|y\|\leq 1}\left(\sum_{j=1}^{n}|\langle x_j,y\rangle|\right),\end{aligned}$$

where for the last equality we used the equality (2.10), which proves the representation (2.2).

Finally, we have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|\leq\sum_{j=1}^{n}|\alpha_j|\max_{j\in\{1,\dots,n\}}|\beta_j|,$$

which implies that

$$\sup_{\|\alpha\|_1\leq 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|\leq\|\beta\|_\infty, \tag{2.11}$$

where $\alpha=(\alpha_1,\dots,\alpha_n)$ and $\beta=(\beta_1,\dots,\beta_n)$.

For $(\beta_1,\dots,\beta_n)\neq 0$, let $j_0\in\{1,...,n\}$ such that $\|\beta\|_\infty=\max_{j\in\{1,...,n\}}|\beta_j|=|\beta_{j_0}|$. Consider $\alpha=(\alpha_1,\dots,\alpha_n)$ with $\alpha_{j_0}=\frac{\overline{\beta_{j_0}}}{|\beta_{j_0}|}$ and $\alpha_j=0$ for $j\neq j_0$. For this choice we get

$$\sum_{j=1}^{n}|\alpha_j|=\frac{\left|\overline{\beta_{j_0}}\right|}{|\beta_{j_0}|}=1 \text{ and } \left|\sum_{j=1}^{n}\alpha_j\beta_j\right|=\left|\frac{\overline{\beta_{j_0}}}{|\beta_{j_0}|}\beta_{j_0}\right|=|\beta_{j_0}|=\|\beta\|_\infty,$$

therefore by (2.11) we obtain the representation

$$\sup_{\|\alpha\|_1\leq 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|=\|\beta\|_\infty \tag{2.12}$$

for any $\beta=(\beta_1,\dots,\beta_n)\in\mathbb{K}^n$.

By taking the supremum in the equality (2.8) and by using the equality (2.12), we have

$$\begin{aligned}\sup_{\|\alpha\|_1\leq 1}\left\|\sum_{j=1}^{n}\alpha_j x_j\right\| &= \sup_{\|\alpha\|_1\leq 1}\left(\sup_{\|y\|\leq 1}\left|\sum_{j=1}^{n}\alpha_j\langle x_j,y\rangle\right|\right)\\ &=\sup_{\|y\|\leq 1}\left(\sup_{\|\alpha\|_1\leq 1}\left|\sum_{j=1}^{n}\alpha_j\langle x_j,y\rangle\right|\right)=\sup_{\|y\|\leq 1}\left(\max_{j\in\{1,...,n\}}|\langle x_j,y\rangle|\right)\\ &=\max_{j\in\{1,...,n\}}\left(\sup_{\|y\|\leq 1}|\langle x_j,y\rangle|\right)=\max_{j\in\{1,...,n\}}\{\|x_j\|\},\end{aligned}$$

which proves (2.3). For the last equality we used the property (2.7). □

Corollary 3.1. *With the assumptions of Theorem 3 we have for $q\geq 1$ that*

$$\frac{1}{n^{1/q}}\|\mathbf{x}\|_{n,q}\leq\|\mathbf{x}\|_{h,n,q}\leq\|\mathbf{x}\|_{n,q} \tag{2.13}$$

for any $\mathbf{x}\in H^n$.

In particular, we have

$$\frac{1}{\sqrt{n}}\|\mathbf{x}\|_2\leq\|\mathbf{x}\|_{h,e}\leq\|\mathbf{x}\|_2 \tag{2.14}$$

for any $\mathbf{x}\in H^n$.

Proof. Let $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and $y \in H$ with $\|y\| \leq 1$, then for $q \geq 1$

$$\left(\sum_{j=1}^{n} |\langle x_j, y\rangle|^q\right)^{1/q} \leq \left(\sum_{j=1}^{n} (\|y\| \|x_i\|)^q\right)^{1/q} = \|y\| \left(\sum_{j=1}^{n} \|x_i\|^q\right)^{1/q} = \|y\| \|\mathbf{x}\|_{n,q}$$

and by taking the supremum over $\|y\| \leq 1$, we get the second inequality in (2.13).

By the properties of complex numbers, we have

$$\max_{j\in\{1,\dots,n\}} \{|\langle x_j, y\rangle|\} \leq \left(\sum_{j=1}^{n} |\langle x_j, y\rangle|^q\right)^{1/q}$$

and by taking the supremum over $\|y\| \leq 1$, we get

$$\sup_{\|y\|\leq 1} \left(\max_{j\in\{1,\dots,n\}} \{|\langle x_j, y\rangle|\}\right) \leq \sup_{\|y\|\leq 1} \left(\sum_{j=1}^{n} |\langle x_j, y\rangle|^q\right) \tag{2.15}$$

and since

$$\begin{aligned} \sup_{\|y\|\leq 1} \left(\max_{j\in\{1,\dots,n\}} \{|\langle x_j, y\rangle|\}\right) &= \max_{j\in\{1,\dots,n\}} \left\{\sup_{\|y\|\leq 1} |\langle x_j, y\rangle|\right\} \\ &= \max_{j\in\{1,\dots,n\}} \{\|x_j\|\} = \|\mathbf{x}\|_{n,\infty}, \end{aligned}$$

then by (2.15) we get

$$\|\mathbf{x}\|_{n,\infty} \leq \|\mathbf{x}\|_{h,n,q} \text{ for any } \mathbf{x} \in H^n. \tag{2.16}$$

Since

$$\left(\sum_{j=1}^{n} \|x_j\|^q\right)^{1/q} \leq \left(n \|\mathbf{x}\|_{n,\infty}^q\right)^{1/q} = n^{1/q} \|\mathbf{x}\|_{n,\infty}$$

then also

$$\frac{1}{n^{1/q}} \|\mathbf{x}\|_{n,q} \leq \|\mathbf{x}\|_{n,\infty} \text{ for any } \mathbf{x} \in H^n. \tag{2.17}$$

By utilising the inequalities (2.16) and (2.17) we obtain the first inequality in (2.13). □

Remark 2.1: In the case of inner product spaces the inequality (2.14) has been obtained in a different and more difficult way in [9] by employing the rotation-invariant normalised positive Borel measure on the unit sphere.

Corollary 3.2. *With the assumptions of Theorem 3 we have for $r \geq q \geq 1$ that*

$$\|\mathbf{x}\|_{h,n,r} \leq \|\mathbf{x}\|_{h,n,q} \leq n^{\frac{r-q}{rq}} \|\mathbf{x}\|_{h,n,r} \tag{2.18}$$

for any any $\mathbf{x} \in H^n$.

In particular, for $q \geq 2$ we have

$$\|\mathbf{x}\|_{h,n,q} \leq \|\mathbf{x}\|_{h,e} \leq n^{\frac{q-2}{2q}} \|\mathbf{x}\|_{h,n,q} \tag{2.19}$$

and for $1 \leq q \leq 2$ we have

$$\|\mathbf{x}\|_{h,e} \leq \|\mathbf{x}\|_{h,n,q} \leq n^{\frac{2-q}{2q}} \|\mathbf{x}\|_{h,e} \tag{2.20}$$

for any $\mathbf{x} \in H^n$.

Proof. We use the following elementary inequalities for the nonnegative numbers a_j, $j = 1, ..., n$ and $r \geq q > 0$ (see for instance [16])

$$\left(\sum_{j=1}^{n} a_j^r\right)^{1/r} \leq \left(\sum_{j=1}^{n} a_j^q\right)^{1/q} \leq n^{\frac{r-q}{rq}} \left(\sum_{j=1}^{n} a_j^r\right)^{1/r}. \tag{2.21}$$

Let $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$ and $y \in H$ with $\|y\| \leq 1$, then for $r \geq q \geq 1$ we have

$$\left(\sum_{j=1}^{n} |\langle x_j, y\rangle|^r\right)^{1/r} \leq \left(\sum_{j=1}^{n} |\langle x_j, y\rangle|^q\right)^{1/q} \leq n^{\frac{r-q}{rq}} \left(\sum_{j=1}^{n} |\langle x_j, y\rangle|^r\right)^{1/r}. \tag{2.22}$$

By taking the supremum over $y \in H$ with $\|y\| \leq 1$ and using Theorem 3, we get (2.18). □

Remark 2.2: If we take $q = 1$ in (2.18), then we get for $r \geq 1$ that

$$\|\mathbf{x}\|_{h,n,r} \leq \|\mathbf{x}\|_{h,n,1} \leq n^{\frac{r-1}{r}} \|\mathbf{x}\|_{h,n,r} \tag{2.23}$$

for any $\mathbf{x} \in H^n$.

In particular, for $r = 2$ we get

$$\|\mathbf{x}\|_{h,e} \leq \|\mathbf{x}\|_{h,n,1} \leq \sqrt{n} \|\mathbf{x}\|_{h,e} \tag{2.24}$$

for any $\mathbf{x} \in H^n$.

3. Some Reverse Inequalities

Recall the following additive reverse of Cauchy-Buniakowski-Schwarz inequality [7] (see also [8, Theorem 5. 14])

Lemma 1. *Let a, $A \in \mathbb{R}$ and $\mathbf{z} = (z_1, \dots, z_n)$, $\mathbf{y} = (y_1, \dots, y_n)$ be two sequences of real numbers with the property that:*

$$ay_j \leq z_j \leq Ay_j \text{ for each } j \in \{1, \dots, n\}. \tag{3.1}$$

Then for any $\mathbf{w} = (w_1, \dots, w_n)$ a sequence of positive real numbers, one has the inequality

$$0 \leq \sum_{j=1}^{n} w_j z_j^2 \sum_{j=1}^{n} w_j y_j^2 - \left(\sum_{j=1}^{n} w_j z_j y_j \right)^2 \leq \frac{1}{4} (A-a)^2 \left(\sum_{j=1}^{n} w_j y_j^2 \right)^2. \tag{3.2}$$

The constant $\frac{1}{4}$ is sharp in (3.2).

O. Shisha and B. Mond obtained in 1967 (see [17]) the following counterparts of (CBS)- inequality (see also [8, Theorem 5.20 & 5.21])

Lemma 2. *Assume that $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ are such that there exists a, A, b, B with the property that:*

$$0 \leq a \leq a_j \leq A \text{ and } 0 < b \leq b_j \leq B \text{ for any } j \in \{1, \dots, n\}, \tag{3.3}$$

then we have the inequality

$$\sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 - \left(\sum_{j=1}^{n} a_j b_j \right)^2 \leq \left(\sqrt{\frac{A}{b}} - \sqrt{\frac{a}{B}} \right)^2 \sum_{j=1}^{n} a_j b_j \sum_{j=1}^{n} b_j^2, \tag{3.4}$$

and

Lemma 3. *Assume that $\mathbf{a}$, $\mathbf{b}$ are nonnegative sequences and there exists γ, Γ with the property that*

$$0 \leq \gamma \leq \frac{a_j}{b_j} \leq \Gamma < \infty \text{ for any } j \in \{1, \dots, n\}. \tag{3.5}$$

Then we have the inequality

$$0 \leq \left(\sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 \right)^{\frac{1}{2}} - \sum_{j=1}^{n} a_j b_j \leq \frac{(\Gamma - \gamma)^2}{4(\gamma + \Gamma)} \sum_{j=1}^{n} b_j^2. \tag{3.6}$$

We have the following result:

Theorem 4. *Let $(H, \|\cdot\|)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$. Then we have*

$$0 \leq \|\mathbf{x}\|_{h,e}^2 - \frac{1}{n} \|\mathbf{x}\|_{h,n,1}^2 \leq \frac{1}{4} n \|\mathbf{x}\|_{n,\infty}^2, \tag{3.7}$$

$$0 \leq \|\mathbf{x}\|_{h,e}^2 - \frac{1}{n} \|\mathbf{x}\|_{h,n,1}^2 \leq \|\mathbf{x}\|_{h,n,1} \|\mathbf{x}\|_{n,\infty} \tag{3.8}$$

and

$$0 \leq \|\mathbf{x}\|_{h,e} - \frac{1}{\sqrt{n}} \|\mathbf{x}\|_{h,n,1} \leq \frac{1}{4}\sqrt{n} \|\mathbf{x}\|_{n,\infty} . \tag{3.9}$$

Proof. Let $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and put $R = \max_{j\in\{1,\dots,n\}} \{\|x_j\|\} = \|\mathbf{x}\|_{n,\infty}$. If $y \in H$ with $\|y\| \leq 1$ then $|\langle x_j, y\rangle| \leq \|y\| \|x_j\| \leq R$ for any $j \in \{1, ..., n\}$.

If we write the inequality (3.2) for $z_j = |\langle x_j, y\rangle|$, $w_j = y_j = 1$, $A = R$ and $a = 0$, we get

$$0 \leq n \sum_{j=1}^{n} |\langle x_j, y\rangle|^2 - \left(\sum_{j=1}^{n} |\langle x_j, y\rangle|\right)^2 \leq \frac{1}{4} n^2 R^2$$

for any $y \in H$ with $\|y\| \leq 1$.

This implies that

$$\sum_{j=1}^{n} |\langle x_j, y\rangle|^2 \leq \frac{1}{n} \left(\sum_{j=1}^{n} |\langle x_j, y\rangle|\right)^2 + \frac{1}{4} n R^2 \tag{3.10}$$

for any $y \in H$ with $\|y\| \leq 1$.

By taking the supremum in (3.10) over $y \in H$ with $\|y\| \leq 1$ we get (3.7).

If we write the inequality (3.4) for $a_j = |\langle x_j, y\rangle|$, $b_j = 1$, $b = B = 1$, $a = 0$ and $A = R$, then we get

$$0 \leq n \sum_{j=1}^{n} |\langle x_j, y\rangle|^2 - \left(\sum_{j=1}^{n} |\langle x_j, y\rangle|\right)^2 \leq nR \sum_{j=1}^{n} |\langle x_j, y\rangle| ,$$

for any $y \in H$ with $\|y\| \leq 1$.

This implies that

$$\sum_{j=1}^{n} |\langle x_j, y\rangle|^2 \leq \frac{1}{n} \left(\sum_{j=1}^{n} |\langle x_j, y\rangle|\right)^2 + R \sum_{j=1}^{n} |\langle x_j, y\rangle| , \tag{3.11}$$

for any $y \in H$ with $\|y\| \leq 1$.

By taking the supremum in (3.11) over $y \in H$ with $\|y\| \leq 1$ we get (3.8).

Finally, if we write the inequality (3.6) for $a_j = |\langle x_j, y\rangle|$, $b_j = 1$, $b = B = 1$, $\gamma = 0$ and $\Gamma = R$, then we have

$$0 \leq \left(n \sum_{j=1}^{n} |\langle x_j, y\rangle|^2\right)^{\frac{1}{2}} - \sum_{j=1}^{n} |\langle x_j, y\rangle| \leq \frac{1}{4} n R,$$

for any $y \in H$ with $\|y\| \leq 1$.

This implies that

$$\left(\sum_{j=1}^{n}|\langle x_j,y\rangle|^2\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{n}}\sum_{j=1}^{n}|\langle x_j,y\rangle| + \frac{1}{4}\sqrt{n}R, \tag{3.12}$$

for any $y \in H$ with $\|y\| \leq 1$.

By taking the supremum in (3.12) over $y \in H$ with $\|y\| \leq 1$ we get (3.9). □

Further, we recall the *Čebyšev's inequality* for *synchronous* n-tuples of vectors $\mathbf{a} = (a_1,\ldots,a_n)$ and $\mathbf{b} = (b_1,\ldots,b_n)$, namely if $(a_j - a_k)(b_j - b_k) \geq 0$ for any $j, k \in \{1,...,n\}$, then

$$\frac{1}{n}\sum_{j=1}^{n}a_jb_j \geq \frac{1}{n}\sum_{j=1}^{n}a_j\frac{1}{n}\sum_{j=1}^{n}b_j. \tag{3.13}$$

In 1950, Biernacki et al. [2] obtained the following discrete version of Grüss' inequality:

Lemma 4. *Assume that* $\mathbf{a} = (a_1,\ldots,a_n)$ *and* $\mathbf{b} = (b_1,\ldots,b_n)$ *are such that there exists real numbers* a, A, b, B *with the property that:*

$$a \leq a_j \leq A \quad \text{and} \quad b \leq b_j \leq B \quad \text{for any } j \in \{1,\ldots,n\}. \tag{3.14}$$

Then

$$\begin{aligned} &\left|\frac{1}{n}\sum_{j=1}^{n}a_jb_j - \frac{1}{n}\sum_{j=1}^{n}a_j\frac{1}{n}\sum_{j=1}^{n}b_j\right| \qquad (3.15)\\ &\leq \frac{1}{n}\left\lceil\frac{n}{2}\right\rceil\left(1-\frac{1}{n}\left\lceil\frac{n}{2}\right\rceil\right)(A-a)(B-b)\\ &= \frac{1}{n^2}\left\lceil\frac{n^2}{4}\right\rceil(A-a)(B-a) \leq \frac{1}{4}(A-a)(B-b), \end{aligned}$$

where $\lceil x\rceil$ *gives the largest integer less than or equal to* x.

The following result also holds:

Theorem 5. *Let* $(H,\|\cdot\|)$ *be an inner product space over the real or complex number field* $\mathbb{K}$ *and* $\mathbf{x} \in H^n$ *with* $\mathbf{x} = (x_1,\ldots,x_n)$. *Then for* q, $r \geq 1$ *we have*

$$\begin{aligned} \|\mathbf{x}\|_{h,n,q+r}^{q+r} &\leq \frac{1}{n}\|\mathbf{x}\|_{h,n,q}^{q}\|\mathbf{x}\|_{h,n,r}^{r} + \frac{1}{n}\left\lceil\frac{n^2}{4}\right\rceil\|\mathbf{x}\|_{n,\infty}^{q+r} \qquad (3.16)\\ &\leq \frac{1}{n}\|\mathbf{x}\|_{h,n,q}^{q}\|\mathbf{x}\|_{h,n,r}^{r} + \frac{1}{4}n\|\mathbf{x}\|_{n,\infty}^{q+r}. \end{aligned}$$

Proof. Let $\mathbf{x}\in H^n$ with $\mathbf{x}=(x_1,\dots,x_n)$ and put $R=\max_{j\in\{1,\dots,n\}}\{\|x_j\|\}=\|\mathbf{x}\|_{n,\infty}$. If $y\in H$ with $\|y\|\le 1$ then $|\langle x_j,y\rangle|\le\|y\|\,\|x_j\|\le R$ for any $j\in\{1,...,n\}$.

If we take into the inequality (3.15) $a_j=|\langle x_j,y\rangle|^q$, $b_j=|\langle x_j,y\rangle|^r$, $a=0$, $A=R^q$, $b=0$ and $B=R^r$, then we get

$$\left|\frac{1}{n}\sum_{j=1}^{n}|\langle x_j,y\rangle|^{q+r}-\frac{1}{n}\sum_{j=1}^{n}|\langle x_j,y\rangle|^{q}\frac{1}{n}\sum_{j=1}^{n}|\langle x_j,y\rangle|^{r}\right|\le\frac{1}{n^2}\left\lceil\frac{n^2}{4}\right\rceil R^{q+r}. \tag{3.17}$$

On the other hand, since the sequences $\{a_j\}_{j=1,\dots,n}$, $\{b_j\}_{j=1,\dots,n}$ are synchronous, then by (3.13) we have

$$0\le\frac{1}{n}\sum_{j=1}^{n}|\langle x_j,y\rangle|^{q+r}-\frac{1}{n}\sum_{j=1}^{n}|\langle x_j,y\rangle|^{q}\frac{1}{n}\sum_{j=1}^{n}|\langle x_j,y\rangle|^{r}.$$

Using (3.17) we then get

$$\sum_{j=1}^{n}|\langle x_j,y\rangle|^{q+r}\le\frac{1}{n}\sum_{j=1}^{n}|\langle x_j,y\rangle|^{q}\sum_{j=1}^{n}|\langle x_j,y\rangle|^{r}+\frac{1}{n}\left\lceil\frac{n^2}{4}\right\rceil R^{q+r} \tag{3.18}$$

for any $y\in H$ with $\|y\|\le 1$.

By taking the supremum in (3.18), we get

$$\begin{aligned}&\sup_{\|y\|\le 1}\left\{\sum_{j=1}^{n}|\langle x_j,y\rangle|^{q+r}\right\}\\&\le\frac{1}{n}\sup_{\|y\|\le 1}\left\{\sum_{j=1}^{n}|\langle x_j,y\rangle|^{q}\sum_{j=1}^{n}|\langle x_j,y\rangle|^{r}\right\}+\frac{1}{n}\left\lceil\frac{n^2}{4}\right\rceil R^{q+r}\\&\le\frac{1}{n}\sup_{\|y\|\le 1}\left\{\sum_{j=1}^{n}|\langle x_j,y\rangle|^{q}\right\}\sup_{\|y\|\le 1}\left\{\sum_{j=1}^{n}|\langle x_j,y\rangle|^{r}\right\}+\frac{1}{n}\left\lceil\frac{n^2}{4}\right\rceil R^{q+r},\end{aligned}$$

which proves the first inequality in (3.16).

The second part of (3.16) is obvious. □

Corollary 5.1. *With the assumptions of Theorem 5 and if $r\ge 1$, then we have*

$$\|\mathbf{x}\|_{h,n,2r}^{2r}\le\frac{1}{n}\|\mathbf{x}\|_{h,n,r}^{2r}+\frac{1}{n}\left\lceil\frac{n^2}{4}\right\rceil\|\mathbf{x}\|_{n,\infty}^{2r}\le\frac{1}{n}\|\mathbf{x}\|_{h,n,r}^{2r}+\frac{1}{4}n\|\mathbf{x}\|_{n,\infty}^{2r}. \tag{3.19}$$

In particular, for $r=1$ we get

$$\|\mathbf{x}\|_{h,e}^{2}\le\frac{1}{n}\|\mathbf{x}\|_{h,n,1}^{2}+\frac{1}{n}\left\lceil\frac{n^2}{4}\right\rceil\|\mathbf{x}\|_{n,\infty}^{2}\le\frac{1}{n}\|\mathbf{x}\|_{h,n,1}^{2}+\frac{1}{4}n\|\mathbf{x}\|_{n,\infty}^{2}. \tag{3.20}$$

The first inequality in (3.20) is better than the second inequality in (3.7).

For an n-tuple of complex numbers $\mathbf{a}=(a_1,\dots,a_n)$ with $n\geq 2$ consider the $(n-1)$-tuple built by the aid of forward differences $\Delta\mathbf{a}=(\Delta a_1,...,\Delta a_{n-1})$ where $\Delta a_k:=a_{k+1}-a_k$ where $k\in\{1,...,n-1\}$. Similarly, if $\mathbf{x}=(x_1,\dots,x_n)\in H^n$ is an n-tuple of vectors we also can consider in a similar way the $(n-1)$-tuple $\Delta\mathbf{x}=(\Delta x_1,...,\Delta x_{n-1})$.

We obtained the following Grüss' type inequalities in terms of forward differences:

Lemma 5. *Assume that* $\mathbf{a}=(a_1,\dots,a_n)$ *and* $\mathbf{b}=(b_1,\dots,b_n)$ *are* n*-tuples of complex numbers. Then*

$$\left|\frac{1}{n}\sum_{j=1}^{n}a_jb_j-\frac{1}{n}\sum_{j=1}^{n}a_j\frac{1}{n}\sum_{j=1}^{n}b_j\right| \tag{3.21}$$
$$\leq\begin{cases}\frac{1}{12}\left(n^2-1\right)\|\Delta\mathbf{a}\|_{n-1,\infty}\|\Delta\mathbf{b}\|_{n-1,\infty}, & [12],\\[2ex] \frac{1}{6}\frac{n^2-1}{n}\|\Delta\mathbf{a}\|_{n-1,\alpha}\|\Delta\mathbf{b}\|_{n-1,\beta}\ \textit{where } \alpha,\ \beta>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, & [5],\\[2ex] \frac{1}{2}\left(1-\frac{1}{n}\right)\|\Delta\mathbf{a}\|_{n-1,1}\|\Delta\mathbf{b}\|_{n-1,1}, & [6].\end{cases}$$

The constants $\frac{1}{12}$, $\frac{1}{6}$ *and* $\frac{1}{2}$ *are best possible in (3.21).*

The following result also holds:

Theorem 6. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$ *and* $\mathbf{x}\in H^n$ *with* $\mathbf{x}=(x_1,\dots,x_n)$. *Then for* $q,\ r\geq 1$ *we have*

$$\|\mathbf{x}\|_{h,n,q+r}^{q+r}\leq\frac{1}{n}\|\mathbf{x}\|_{h,n,q}^{q}\|\mathbf{x}\|_{h,n,r}^{r} \tag{3.22}$$
$$+\begin{cases}\frac{1}{12}qr\left(n^2-1\right)n\|\mathbf{x}\|_{n,\infty}^{q+r-2}\|\Delta\mathbf{x}\|_{n-1,\infty}^{2},\\[2ex] \frac{1}{6}\left(n^2-1\right)qr\|\mathbf{x}\|_{n,\infty}^{q+r-2}\|\Delta\mathbf{x}\|_{h,n-1,\alpha}\|\Delta\mathbf{x}\|_{h,n-1,\beta}\\ \textit{where } \alpha,\ \beta>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1,\\[2ex] \frac{1}{2}\left(n-1\right)qr\|\mathbf{x}\|_{n,\infty}^{q+r-2}\|\Delta\mathbf{x}\|_{h,n-1,1}^{2}.\end{cases}$$

Proof. Let $\mathbf{x}\in H^n$ with $\mathbf{x}=(x_1,\dots,x_n)$ and $y\in H$ with $\|y\|\leq 1$. If we take into the inequality (3.21) $a_j=|\langle x_j,y\rangle|^q$, $b_j=|\langle x_j,y\rangle|^r$, then we get

$$\left|\frac{1}{n}\sum_{j=1}^{n}|\langle x_j,y\rangle|^{q+r}-\frac{1}{n}\sum_{j=1}^{n}|\langle x_j,y\rangle|^{q}\frac{1}{n}\sum_{j=1}^{n}|\langle x_j,y\rangle|^{r}\right| \tag{3.23}$$

$$\leq \begin{cases} \frac{1}{12}\left(n^2-1\right)\max_{j=1,\dots,n-1}\left|\Delta\left|\langle x_j,y\rangle\right|^q\right|\max_{j=1,\dots,n-1}\left|\Delta\left|\langle x_j,y\rangle\right|^r\right|, \\ \\ \frac{1}{6}\frac{n^2-1}{n}\left(\sum_{j=1}^{n-1}\left|\Delta\left|\langle x_j,y\rangle\right|^q\right|^\alpha\right)^{1/\alpha}\left(\sum_{j=1}^{n-1}\left|\Delta\left|\langle x_j,y\rangle\right|^r\right|^\beta\right)^{1/\beta} \\ \text{where } \alpha,\ \beta>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \\ \frac{1}{2}\left(1-\frac{1}{n}\right)\sum_{j=1}^{n-1}\left|\Delta\left|\langle x_j,y\rangle\right|^q\right|\sum_{j=1}^{n-1}\left|\Delta\left|\langle x_j,y\rangle\right|^r\right|. \end{cases}$$

We use the following elementary inequality for powers $p\geq 1$

$$|a^p-b^p|\leq pR^{p-1}|a-b|$$

where $a,\,b\in[0,R]$.

Put $R=\max_{j\in\{1,\dots,n\}}\{\|x_j\|\}=\|\mathbf{x}\|_{n,\infty}$. Then for any $y\in H$ with $\|y\|\leq 1$ we have $|\langle x_j,y\rangle|\leq\|y\|\,\|x_j\|\leq R$ for any $j\in\{1,\dots,n\}$.

Therefore

$$\begin{aligned} \left|\Delta\left|\langle x_j,y\rangle\right|^q\right| &= \left|\left|\langle x_{j+1},y\rangle\right|^q-\left|\langle x_j,y\rangle\right|^q\right|\leq qR^{q-1}\left|\left|\langle x_{j+1},y\rangle\right|-\left|\langle x_j,y\rangle\right|\right| \\ &\leq qR^{q-1}\left|\langle x_{j+1},y\rangle-\langle x_j,y\rangle\right|=qR^{q-1}\left|\langle\Delta x_j,y\rangle\right| \end{aligned} \tag{3.24}$$

for any $j=1,\dots,n-1$, where $\Delta x_j=x_{j+1}-x_j$ is the forward difference.

On the other hand, since the sequences $\{a_j\}_{j=1,\dots,n}$, $\{b_j\}_{j=1,\dots,n}$ are synchronous, then we have

$$0\leq\frac{1}{n}\sum_{j=1}^{n}\left|\langle x_j,y\rangle\right|^{q+r}-\frac{1}{n}\sum_{j=1}^{n}\left|\langle x_j,y\rangle\right|^q\frac{1}{n}\sum_{j=1}^{n}\left|\langle x_j,y\rangle\right|^r \tag{3.25}$$

and by the first inequality in (3.23) we get

$$\begin{aligned} &\sum_{j=1}^{n}\left|\langle x_j,y\rangle\right|^{q+r} \\ &\leq\frac{1}{n}\sum_{j=1}^{n}\left|\langle x_j,y\rangle\right|^q\sum_{j=1}^{n}\left|\langle x_j,y\rangle\right|^r \\ &+\frac{1}{12}\left(n^2-1\right)nqR^{q-1}\max_{j=1,\dots,n-1}\left|\langle\Delta x_j,y\rangle\right|rR^{r-1}\max_{j=1,\dots,n-1}\left|\langle\Delta x_j,y\rangle\right| \\ &=\frac{1}{n}\sum_{j=1}^{n}\left|\langle\Delta x_j,y\rangle\right|^q\sum_{j=1}^{n}\left|\langle\Delta x_j,y\rangle\right|^r \\ &+\frac{1}{12}\left(n^2-1\right)nqrR^{q+r-2}\left(\max_{j=1,\dots,n-1}\left|\langle\Delta x_j,y\rangle\right|\right)^2 \end{aligned} \tag{3.26}$$

for any $y\in H$ with $\|y\|\leq 1$.

Taking the supremum over $y \in H$ with $\|y\| \leq 1$ in (3.26) we get the first branch in the inequality (3.22).

We also have, by (3.24), that

$$\left(\sum_{j=1}^{n-1} \left|\Delta \left|\langle x_j, y\rangle\right|^q\right|^\alpha\right)^{1/\alpha} \leq \left[\left(qR^{q-1}\right)^\alpha \sum_{j=1}^{n-1} \left|\langle \Delta x_j, y\rangle\right|^\alpha\right]^{1/\alpha} = qR^{q-1}\left(\sum_{j=1}^{n-1} \left|\langle \Delta x_j, y\rangle\right|^\alpha\right)^{1/\alpha}$$

and, similarly,

$$\left(\sum_{j=1}^{n-1} \left|\Delta \left|\langle x_j, y\rangle\right|^r\right|^\beta\right)^{1/\beta} \leq rR^{r-1}\left(\sum_{j=1}^{n-1} \left|\langle \Delta x_j, y\rangle\right|^\beta\right)^{1/\beta}$$

where $\alpha,\ \beta > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1$.

By the second inequality in (3.23) and by (3.25) we have

$$\begin{aligned} &\sum_{j=1}^{n} \left|\langle x_j, y\rangle\right|^{q+r} \qquad (3.27)\\ &\leq \frac{1}{n}\sum_{j=1}^{n} \left|\langle x_j, y\rangle\right|^q \sum_{j=1}^{n} \left|\langle x_j, y\rangle\right|^r \\ &+ \frac{1}{6}\left(n^2-1\right)\left(\sum_{j=1}^{n-1} \left|\Delta \left|\langle x_j, y\rangle\right|^q\right|^\alpha\right)^{1/\alpha}\left(\sum_{j=1}^{n-1} \left|\Delta \left|\langle x_j, y\rangle\right|^r\right|^\beta\right)^{1/\beta} \\ &\leq \frac{1}{n}\sum_{j=1}^{n} \left|\langle x_j, y\rangle\right|^q \sum_{j=1}^{n} \left|\langle x_j, y\rangle\right|^r \\ &+ \frac{1}{6}\left(n^2-1\right) qrR^{q+r-2}\left(\sum_{j=1}^{n-1} \left|\langle \Delta x_j, y\rangle\right|^\alpha\right)^{1/\alpha}\left(\sum_{j=1}^{n-1} \left|\langle \Delta x_j, y\rangle\right|^\beta\right)^{1/\beta} \end{aligned}$$

for any $y \in H$ with $\|y\| \leq 1$, where $\alpha,\ \beta > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Taking the supremum over $y \in H$ with $\|y\| \leq 1$ in (3.27) we get the second branch in the inequality (3.22).

We also have, by (3.24), that

$$\sum_{j=1}^{n-1} \left|\Delta \left|\langle x_j, y\rangle\right|^q\right| \leq qR^{q-1} \sum_{j=1}^{n-1} \left|\langle \Delta x_j, y\rangle\right|$$

and

$$\sum_{j=1}^{n-1} \left|\Delta \left|\langle x_j, y\rangle\right|^r\right| \leq rR^{r-1} \sum_{j=1}^{n-1} \left|\langle \Delta x_j, y\rangle\right|.$$

By the third inequality in (3.23) and by (3.25) we have

$$\begin{aligned}
\sum_{j=1}^{n} \left|\langle x_j, y\rangle\right|^{q+r} &\leq \frac{1}{n} \sum_{j=1}^{n} \left|\langle x_j, y\rangle\right|^q \sum_{j=1}^{n} \left|\langle x_j, y\rangle\right|^r \\
&\quad + \frac{1}{2}(n-1) \sum_{j=1}^{n-1} \left|\Delta \left|\langle x_j, y\rangle\right|^q\right| \sum_{j=1}^{n-1} \left|\Delta \left|\langle x_j, y\rangle\right|^r\right| \\
&\leq \frac{1}{n} \sum_{j=1}^{n} \left|\langle x_j, y\rangle\right|^q \sum_{j=1}^{n} \left|\langle x_j, y\rangle\right|^r \\
&\quad + \frac{1}{2}(n-1) qrR^{q+r-2} \sum_{j=1}^{n-1} \left|\langle \Delta x_j, y\rangle\right| \sum_{j=1}^{n-1} \left|\langle \Delta x_j, y\rangle\right|
\end{aligned} \tag{3.28}$$

for any $y \in H$ with $\|y\| \leq 1$.

Taking the supremum over $y \in H$ with $\|y\| \leq 1$ in (3.28) we get the third branch in the inequality (3.22). □

Corollary 6.1. *With the assumptions of Theorem 6 and if $r \geq 1$, then we have*

$$\|\mathbf{x}\|_{h,n,2r}^{2r} \leq \frac{1}{n} \|\mathbf{x}\|_{h,n,r}^{2r} + \begin{cases} \frac{1}{12} r^2 \left(n^2-1\right) n \|\mathbf{x}\|_{n,\infty}^{2r-2} \|\Delta\mathbf{x}\|_{n-1,\infty}^2, \\ \\ \frac{1}{6} r^2 \left(n^2-1\right) \|\mathbf{x}\|_{n,\infty}^{2r-2} \|\Delta\mathbf{x}\|_{h,n-1,\alpha} \|\Delta\mathbf{x}\|_{h,n-1,\beta} \\ \text{where } \alpha,\ \beta > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \\ \frac{1}{2} r^2 (n-1) \|\mathbf{x}\|_{n,\infty}^{2r-2} \|\Delta\mathbf{x}\|_{h,n-1,1}^2. \end{cases} \tag{3.29}$$

In particular, for $r = 1$ we get

$$\|\mathbf{x}\|_{h,e}^2 \leq \frac{1}{n} \|\mathbf{x}\|_{h,n,1}^2 + \begin{cases} \frac{1}{12} \left(n^2-1\right) n \|\Delta\mathbf{x}\|_{n-1,\infty}^2, \\ \\ \frac{1}{6} \left(n^2-1\right) \|\Delta\mathbf{x}\|_{h,n-1,\alpha} \|\Delta\mathbf{x}\|_{h,n-1,\beta} \\ \text{where } \alpha,\ \beta > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \\ \frac{1}{2} (n-1) \|\Delta\mathbf{x}\|_{h,n-1,1}^2. \end{cases} \tag{3.30}$$

4. Further Inequalities

In 1992, J. Pečarić [15] proved the following general inequality in inner product spaces:

Lemma 6. *Let* $y,\ x_1,\dots,\ x_n \in H$ *and* $c_1,\dots,\ c_n \in \mathbb{K}$. *Then*

$$\left|\sum_{i=1}^{n} c_i \langle x_i, y\rangle\right|^2 \leq \|y\|^2 \sum_{i=1}^{n} |c_i|^2 \left(\sum_{j=1}^{n} |\langle x_i, x_j\rangle|\right) \leq \|y\|^2 \sum_{i=1}^{n} |c_i|^2 \max_{1\leq i\leq n}\left\{\sum_{j=1}^{n} |\langle x_i, x_j\rangle|\right\}. \tag{4.1}$$

He showed that the Bombieri inequality [4] may be obtained from (4.1) for the choice $c_i = \overline{\langle x_i, y\rangle}$ (using the second inequality), the Selberg inequality [14, p. 394] may be obtained from the first part of (4.1) for the choice

$$c_i = \frac{\overline{\langle x_i, y\rangle}}{\sum_{j=1}^{n} |\langle x_i, x_j\rangle|}, \quad i \in \{1,\dots,n\};$$

while the Heilbronn inequality [13] may be obtained from the first part of (4.1) if one chooses $c_i = \frac{\overline{\langle x_i, y\rangle}}{|\langle x_i, y\rangle|}$, for any $i \in \{1,\dots,n\}$.

Theorem 7. *Let* $(H, \langle\cdot,\cdot\rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$ *and* $\mathbf{x} \in H^n$ *with* $\mathbf{x} = (x_1,\dots,x_n)$. *Then for* $q \geq 1$ *we have*

$$\|\mathbf{x}\|_{h,n,q} \leq \|\mathbf{x}\|_{h,n,2(q-1)}^{1-1/q} \left[\max_{1\leq i\leq n}\left\{\sum_{j=1}^{n} |\langle x_i, x_j\rangle|\right\}\right]^{\frac{1}{2q}}. \tag{4.2}$$

In particular, for $q = 1$ *we get*

$$\|\mathbf{x}\|_{h,n,1} \leq \left[\max_{1\leq i\leq n}\left\{\sum_{j=1}^{n} |\langle x_i, x_j\rangle|\right\}\right]^{\frac{1}{2}} \tag{4.3}$$

while for $q = 2$ *we get*

$$\|\mathbf{x}\|_{h,e} \leq \left[\max_{1\leq i\leq n}\left\{\sum_{j=1}^{n} |\langle x_i, x_j\rangle|\right\}\right]^{\frac{1}{2}}. \tag{4.4}$$

Proof. If we take in (4.1) $c_i = \overline{\langle x_i, y\rangle}\, |\langle x_i, y\rangle|^{q-2}$, $i \in \{1,...,n\}$, then we get

$$\left(\sum_{i=1}^{n} |\langle x_i, y\rangle|^q\right)^2 \leq \|y\|^2 \sum_{i=1}^{n} |\langle x_i, y\rangle|^{2(q-1)} \max_{1\leq i\leq n}\left\{\sum_{j=1}^{n} |\langle x_i, x_j\rangle|\right\} \tag{4.5}$$

for any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and $y \in H$.

By taking the square root in (4.5) we get

$$\sum_{i=1}^{n} |\langle x_i, y\rangle|^q \tag{4.6}$$
$$\leq \|y\| \left(\sum_{i=1}^{n} |\langle x_i, y\rangle|^{2(q-1)}\right)^{1/2} \left[\max_{1\leq i\leq n}\left\{\sum_{j=1}^{n} |\langle x_i, x_j\rangle|\right\}\right]^{1/2},$$

for any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and $y \in H$.

If we take the supremum in (4.6) for $\|y\| \leq 1$, then we get

$$\|\mathbf{x}\|_{h,n,q}^{q} \leq \|\mathbf{x}\|_{h,n,2(q-1)}^{q-1} \left[\max_{1\leq i\leq n}\left\{\sum_{j=1}^{n} |\langle x_i, x_j\rangle|\right\}\right]^{1/2}$$

for any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and $y \in H$.

This proves (4.2). □

In 1941, R. P. Boas [3] and in 1944, independently, R. Bellman [1] proved the following generalization of Bessel's inequality

Lemma 7. *If* $y, x_1, \dots, x_n$ *are elements of an inner product space* $(H; \langle\cdot,\cdot\rangle)$, *then the following inequality:*

$$\sum_{i=1}^{n} |\langle x_i, y\rangle|^2 \leq \|y\|^2 \left[\max_{1\leq i\leq n} \|x_i\|^2 + \left(\sum_{1\leq i\neq j\leq n} |\langle x_i, x_j\rangle|^2\right)^{\frac{1}{2}}\right], \tag{4.7}$$

holds.

A generalization of the Boas-Bellman result was given in Mitrinović-Pečarić-Fink [14, p. 392] where they proved the following:

Lemma 8. *If* y, $x_1, \dots, x_n$ *are as in Lemma 7 and* $c_1, \dots, c_n \in \mathbb{K}$, *then one has the inequality:*

$$\left|\sum_{i=1}^{n} c_i \langle x_i, y\rangle\right|^2 \leq \|y\|^2 \sum_{i=1}^{n} |c_i|^2 \left[\max_{1\leq i\leq n} \|x_i\|^2 + \left(\sum_{1\leq i\neq j\leq n} |\langle x_i, x_j\rangle|^2\right)^{\frac{1}{2}}\right]. \tag{4.8}$$

They also noted that if in (4.8) one chooses $c_i = \overline{(x, y_i)}$, then this inequality becomes (4.7).

Using a similar argument to the one in Theorem 7 and Lemma 8 we have:

Theorem 8. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$. Then for $q \geq 1$ we have*

$$\|\mathbf{x}\|_{h,n,q} \leq \|\mathbf{x}\|_{h,n,2(q-1)}^{1-1/q} \left[\max_{1 \leq i \leq n} \|x_i\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle|^2 \right)^{\frac{1}{2}} \right]^{\frac{1}{2q}}. \quad (4.9)$$

In particular, for $q = 1$ we get

$$\|\mathbf{x}\|_{h,n,1} \leq \left[\max_{1 \leq i \leq n} \|x_i\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle|^2 \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \quad (4.10)$$

while for $q = 2$ we get

$$\|\mathbf{x}\|_{h,e} \leq \left[\max_{1 \leq i \leq n} \|x_i\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle|^2 \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}. \quad (4.11)$$

In [10] we obtained the following result that provides some companions to the Boas-Bellman inequality above:

Lemma 9. *Let $y, x_1, \dots, x_n \in H$ and $c_1, \dots, c_n \in \mathbb{K}$. Then*

$$\left| \sum_{i=1}^{n} c_i \langle x_i, y \rangle \right|^2 \quad (4.12)$$

$$\leq \|y\|^2 \times \begin{cases} \max\limits_{1 \leq i \leq n} \left\{ |c_i|^2 \right\} \left\{ \sum_{i=1}^{n} \|x_i\|^2 + \sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle| \right\}; \\ \\ \left(\sum_{i=1}^{n} |c_i|^{2p} \right)^{\frac{1}{p}} \left\{ \left(\sum_{i=1}^{n} \|x_i\|^{2r} \right)^{\frac{1}{r}} \right. \\ \left. + (n-1)^{\frac{1}{p}} \left(\sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle|^r \right)^{\frac{1}{r}} \right\}, \\ \text{where } \; p > 1, \frac{1}{p} + \frac{1}{r} = 1; \\ \\ \sum_{i=1}^{n} |c_i|^2 \left\{ \max\limits_{1 \leq i \leq n} \|x_i\|^2 + (n-1) \max\limits_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle| \right\}. \end{cases}$$

We have:

Theorem 9. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$. Then for $q \geq 1$ we* *have*

$$\|\mathbf{x}\|_{h,n,q} \leq \begin{cases} \max\limits_{1\leq i\leq n}\{\|x_i\|\}^{1-1/q}\left\{\sum_{i=1}^n \|x_i\|^2 + \sum_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|\right\}^{\frac{1}{2q}}; \\ \\ \|\mathbf{x}\|_{h,n,2p(q-1)}^{1-1/q}\left\{\left(\sum_{i=1}^n \|x_i\|^{2r}\right)^{\frac{1}{r}} \right. \\ \left. +\,(n-1)^{\frac{1}{p}}\left(\sum_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|^r\right)^{\frac{1}{r}}\right\}^{\frac{1}{2q}}, \\ \text{where } \ p>1, \frac{1}{p}+\frac{1}{r}=1; \\ \\ \|\mathbf{x}\|_{h,n,2(q-1)}^{1-1/q}\left\{\max\limits_{1\leq i\leq n}\|x_i\|^2 + (n-1)\max\limits_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|\right\}^{\frac{1}{2q}}. \end{cases} \tag{4.13}$$

In particular, we have for $q=1$ *that*

$$\|\mathbf{x}\|_{h,n,1} \leq \begin{cases} \left\{\sum_{i=1}^n \|x_i\|^2 + \sum_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|\right\}^{\frac{1}{2}}; \\ \\ \left\{\left(\sum_{i=1}^n \|x_i\|^{2r}\right)^{\frac{1}{r}} \right. \\ \left. +\,(n-1)^{\frac{1}{p}}\left(\sum_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|^r\right)^{\frac{1}{r}}\right\}^{\frac{1}{2}}, \\ \text{where } \ p>1, \frac{1}{p}+\frac{1}{r}=1; \\ \\ \left\{\max\limits_{1\leq i\leq n}\|x_i\|^2 + (n-1)\max\limits_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|\right\}^{\frac{1}{2}}, \end{cases} \tag{4.14}$$

while for $q=2$ *that*

$$\|\mathbf{x}\|_{h,e}^2 \leq \begin{cases} \max\limits_{1\leq i\leq n}\{\|x_i\|\}\left\{\sum_{i=1}^n \|x_i\|^2 + \sum_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|\right\}^{\frac{1}{2}}; \\ \\ \|\mathbf{x}\|_{h,n,2p}\left\{\left(\sum_{i=1}^n \|x_i\|^{2r}\right)^{\frac{1}{r}} \right. \\ \left. +\,(n-1)^{\frac{1}{p}}\left(\sum_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|^r\right)^{\frac{1}{r}}\right\}^{\frac{1}{2}}, \\ \text{where } \ p>1, \frac{1}{p}+\frac{1}{r}=1; \\ \\ \|\mathbf{x}\|_{h,e}\left\{\max\limits_{1\leq i\leq n}\|x_i\|^2 + (n-1)\max\limits_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|\right\}^{\frac{1}{2}}. \end{cases} \tag{4.15}$$

Proof. If we take in (4.12) $c_i = \overline{\langle x_i, y\rangle} \left|\langle x_i, y\rangle\right|^{q-2}$, $i \in \{1, ..., n\}$, then we get

$$\left(\sum_{i=1}^{n} \left|\langle x_i, y\rangle\right|^q\right)^2 \leq \|y\|^2 \times \begin{cases} \max\limits_{1\leq i\leq n}\left\{\left|\langle x_i, y\rangle\right|^{2(q-1)}\right\}\left\{\sum_{i=1}^{n}\|x_i\|^2 + \sum_{1\leq i\neq j\leq n}\left|\langle x_i, x_j\rangle\right|\right\}; \\ \left(\sum_{i=1}^{n}\left|\langle x_i, y\rangle\right|^{2p(q-1)}\right)^{\frac{1}{p}}\left\{\left(\sum_{i=1}^{n}\|x_i\|^{2r}\right)^{\frac{1}{r}} \right. \\ \left. + (n-1)^{\frac{1}{p}}\left(\sum_{1\leq i\neq j\leq n}\left|\langle x_i, x_j\rangle\right|^r\right)^{\frac{1}{r}}\right\}, \\ \text{where } \ p > 1, \frac{1}{p} + \frac{1}{r} = 1; \\ \sum_{i=1}^{n}\left|\langle x_i, y\rangle\right|^{2(q-1)}\left\{\max\limits_{1\leq i\leq n}\|x_i\|^2 + (n-1)\max\limits_{1\leq i\neq j\leq n}\left|\langle x_i, x_j\rangle\right|\right\}, \end{cases} \tag{4.16}$$

for any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and $y \in H$.

If we take the square root in this inequality, then we get

$$\sum_{i=1}^{n} \left|\langle x_i, y\rangle\right|^q \leq \|y\| \times \begin{cases} \max\limits_{1\leq i\leq n}\left\{\left|\langle x_i, y\rangle\right|^{q-1}\right\}\left\{\sum_{i=1}^{n}\|x_i\|^2 + \sum_{1\leq i\neq j\leq n}\left|\langle x_i, x_j\rangle\right|\right\}^{1/2}; \\ \left(\sum_{i=1}^{n}\left|\langle x_i, y\rangle\right|^{2p(q-1)}\right)^{\frac{1}{2p}}\left\{\left(\sum_{i=1}^{n}\|x_i\|^{2r}\right)^{\frac{1}{r}} \right. \\ \left. + (n-1)^{\frac{1}{p}}\left(\sum_{1\leq i\neq j\leq n}\left|\langle x_i, x_j\rangle\right|^r\right)^{\frac{1}{r}}\right\}^{1/2}, \\ \text{where } \ p > 1, \frac{1}{p} + \frac{1}{r} = 1; \\ \left(\sum_{i=1}^{n}\left|\langle x_i, y\rangle\right|^{2(q-1)}\right)^{1/2}\left\{\max\limits_{1\leq i\leq n}\|x_i\|^2 + (n-1)\max\limits_{1\leq i\neq j\leq n}\left|\langle x_i, x_j\rangle\right|\right\}^{1/2}, \end{cases} \tag{4.17}$$

for any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and $y \in H$.

If we take the supremum over $y \in H$, $\|y\| \leq 1$, then we get

$$\|\mathbf{x}\|_{h,n,q}^{q} \leq \begin{cases} \max\limits_{1\leq i\leq n}\left\{\|x_i\|^{q-1}\right\}\left\{\sum_{i=1}^{n}\|x_i\|^2+\sum_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|\right\}^{1/2}; \\ \\ \|\mathbf{x}\|_{h,n,2p(q-1)}^{q-1}\left\{\left(\sum_{i=1}^{n}\|x_i\|^{2r}\right)^{\frac{1}{r}}\right. \\ \left.+\,(n-1)^{\frac{1}{p}}\left(\sum_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|^r\right)^{\frac{1}{r}}\right\}^{1/2}, \\ \text{where } \ p>1, \frac{1}{p}+\frac{1}{r}=1; \\ \\ \|\mathbf{x}\|_{h,n,2(q-1)}^{q-1}\left\{\max\limits_{1\leq i\leq n}\|x_i\|^2+(n-1)\max\limits_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|\right\}^{1/2}, \end{cases}$$

for any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$, which produces the desired result (4.13). □

The interested reader may obtain other similar results by utilising the following inequalities obtained in [11]

Lemma 10. *Let $y, x_1, \ldots, x_n$ be vectors of an inner product space $(H; \langle\cdot,\cdot\rangle)$ and $c_1, \ldots, c_n \in \mathbb{K}$. Then one has the inequalities:*

$$\left|\sum_{i=1}^{n} c_i \langle x_i, y\rangle\right|^2 \leq \|y\|^2 \times \begin{cases} D \\ E \\ F \end{cases}, \tag{4.18}$$

where

$$D := \begin{cases} \max\limits_{1\leq k\leq n}|c_k|^2\sum\limits_{i,j=1}^{n}|\langle x_i,x_j\rangle|; \\ \\ \max\limits_{1\leq k\leq n}|c_k|\left(\sum\limits_{i=1}^{n}|c_i|^r\right)^{\frac{1}{r}}\left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|\langle x_i,x_j\rangle|\right)^s\right]^{\frac{1}{s}}, \quad r>1,\ \frac{1}{r}+\frac{1}{s}=1; \\ \\ \max\limits_{1\leq k\leq n}|c_k|\sum\limits_{k=1}^{n}|c_k|\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|\langle x_i,x_j\rangle|\right); \end{cases}$$

$$E := \begin{cases} \left(\sum_{k=1}^{n} |c_k|^p\right)^{\frac{1}{p}} \max_{1\le i\le n} |c_i| \left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n} |\langle x_i, x_j\rangle|\right)^q\right)^{\frac{1}{q}}, & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left(\sum_{k=1}^{n} |c_k|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |c_i|^t\right)^{\frac{1}{t}} \left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n} |\langle x_i, x_j\rangle|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}, & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ & t>1,\ \frac{1}{t}+\frac{1}{u}=1; \\ \left(\sum_{k=1}^{n} |c_k|^p\right)^{\frac{1}{p}} \sum_{i=1}^{n} |c_i| \max_{1\le i\le n} \left\{\left(\sum_{j=1}^{n} |\langle x_i, x_j\rangle|^q\right)^{\frac{1}{q}}\right\}, & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \end{cases}$$

and

$$F := \begin{cases} \sum_{k=1}^{n} |c_k| \max_{1\le i\le n} |c_i| \sum_{i=1}^{n} \left[\max_{1\le j\le n} |\langle x_i, x_j\rangle|\right]; \\ \sum_{k=1}^{n} |c_k| \left(\sum_{i=1}^{n} |c_i|^m\right)^{\frac{1}{m}} \left(\sum_{i=1}^{n} \left[\max_{1\le j\le n} |\langle x_i, x_j\rangle|\right]^l\right)^{\frac{1}{l}}, \\ \qquad m>1,\ \frac{1}{m}+\frac{1}{l}=1; \\ \left(\sum_{k=1}^{n} |c_k|\right)^2 \max_{i,1\le j\le n} |\langle x_i, x_j\rangle|. \end{cases}$$

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