BOUNDED SUBSETS OF DISTRIBUTIONS IN D' GENERATED WITH BOUNDARY VALUES OF FUNCTIONS OF THE SPACE $H^p, 1 \leq p < \infty$

Vesna Manova Eraković

Abstract. In this paper it is proved that if a subset L of $H^p(\Pi^+)$, $1 \leq p < \infty$ is bounded in $H^p(\Pi^+)$, then the set of regular distributions in D' generated with the boundary values of the functions of L is also bounded in D'.

Also, in the paper are given some corollaries that follow from the main result.

key words: distribution, bounded set, space H^p, $1 \leq p < \infty$, boundary value of function

0. INTRODUCTION

0.1: Denotations which will be used in the paper

Let Π^+ denote the upper half plane i.e. $\Pi^+ = \{z \in \mathbf{C} \mid Im z > 0\}$.

For a given function f which is analytic on some region Ω we will write $f \in H(\Omega)$.

For a function f, $f\colon \Omega \to \mathbf{C}^n$, $\Omega \subseteq \mathbf{R}^n$ $x \in \Omega$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j \in \mathbf{N} \cup \{0\}$, $j \in \{1, 2, \dots, n\}$, $D_x^\alpha f$ denotes the differential operator

$$D_x^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}, \quad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n .$$

$L^p(\Omega)$ is the Lebesque space of measurable functions f on Ω for which

$$\|f\|_{L^p(\Omega)} = \|f\|_p = \left(\int_\Omega |f(x)|^p dx \right)^{1/p} < \infty, \quad 0 \leq p < \infty,$$

and

$$\|f\|_{L^\infty(\Omega)} = \|f\|_\infty = \operatorname*{supess}_{x \in \Omega} |f(x)| .$$

$L^p_{\rm loc}(\Omega)$ is the space of locally integrable functions on Ω, i.e. $f(x) \in L^p_{\rm loc}(\Omega)$ if $f(x) \in L^p(\Omega')$, for every bounded subregion Ω' of Ω.

0.2. The spaces H^p defined on Π^+ and some of their properties

Let $f(z) \in H(\Pi^+)$ and let $0 < p < \infty$. We say that $f(z) \in H^p(\Pi^+)$ iff

$$\sup_{0<y<\infty} \int_{-\infty}^{\infty} |f(x+iy)|^p dx < \infty .$$

$H^p(\Pi^+)$ is normed space with the following norm

$$\|f\|_{H^p} = \sup_{0<y<\infty} \left(\int_{-\infty}^{\infty} |f(x+iy)|^p dx \right)^{1/p}$$

In the case $p = \infty$, $H^\infty(\Pi^+)$ is the space of all bounded analytic functions on Π^+, for which the norm is defined by

$$\|f\|_{H^\infty} = \sup_{z\in\Pi^+} |f(z)| .$$

It is known [5] that every function $f(z) \in H^p(\Pi^+)$, $0 < p < \infty$, for almost every $x \in \mathbf{R}$, has nontangential limit $f^*(x)$, $f^*(x) \in L^p(\mathbf{R})$ satisfying

$$\int_{-\infty}^{\infty} |f^*(x)|^p dx = \|f\|_p^p = \|f\|_{H^p}^p$$

and

$$\lim_{y\to 0^+} \|f(x+iy) - f^*(x)\|_p^p = 0 .$$

0.3. Some notions of distributions

$C^\infty(\mathbf{R}^n)$ denotes the space of all complex valued infinitely differentiable functions on $\mathbf{R}^n$ and $C_0^\infty(\mathbf{R}^n)$ denotes the subspace of $C^\infty(\mathbf{R}^n)$ that consists of those functions of $C^\infty(\mathbf{R}^n)$ which have compact support. Support of a function f, denoted by supp(f), is the closure of $\{x \mid f(x) \neq 0\}$ in $\mathbf{R}^n$.

$D = D(\mathbf{R}^n)$ denotes the space of $C_0^\infty(\mathbf{R}^n)$ functions in which convergence is defined in the following way: a sequence $\{\varphi_\lambda\}$ of functions $\varphi_\lambda \in D$ converges to $\varphi \in D$ in D as $\lambda \to \lambda_0$ if and only if there is a compact set $K \subset \mathbf{R}^n$ such that $\operatorname{supp}(\varphi_\lambda) \subseteq K$ for each λ, $\operatorname{supp}(\varphi) \subseteq K$ and for every n-tuple α of nonnegative integers the sequence $\{D_t^\alpha \varphi_\lambda(t)\}$ converges to $D_t^\alpha \varphi(t)$ uniformly on K as $\lambda \to \lambda_0$.

$D' = D'(\mathbf{R}^n)$ is the space of all continuous linear functionals on D, where continuity means that $\varphi_\lambda \to \varphi$ in D as $\lambda \to \lambda_0$ implies $\langle T, \varphi_\lambda\rangle \to \langle T, \varphi\rangle$ as $\lambda \to \lambda_0$, for $T \in D'$.

Note: $\langle T, \varphi\rangle$ denotes the value of the functional T, when it acts on the function φ.

D' is called the space of distributions.

Let $\varphi \in D$ and $f(t) \in L^1_{\text{loc}}(\mathbf{R}^n)$. Then the functional T_f from D to $\mathbf{C}$, defined by

$$\langle T_f, \varphi\rangle = \int\limits_{\mathbf{R}^n} f(t)\varphi(t)dt\,, \quad \varphi \in D$$

is distribution on D, called regular distribution generated with f.

A subset B of D is bounded in D iff:

1°. There is a compact subset $K \subset \mathbf{R}^n$, such that $\operatorname{supp}(\varphi) \subseteq K$, for every $\varphi \in B$.

2°. For every n-tuple α of nonnegative integers, there is a constant M_α such that

$$\sup_{t\in K} |D^\alpha \varphi(t)| \le M_\alpha\,, \quad \text{for every} \quad \varphi \in B\,.$$

A subset B' of D' is bounded in D' if there is a constant M_B such that

$$\sup_{\varphi\in B} |\langle T, \varphi\rangle| \le M_B$$

for all $T \in B'$ and for all bounded subsets $B \subset D$.

For $1 \le p < \infty$, $D_{L^p}(\mathbf{R}^n)$ denotes the space of all infinitely differentiable functions φ, such that $D^\beta \varphi(t) \in L^p$, for every n-tuple of nonnegative integers β, and in which the convergence is defined in the following way: a sequence $\{\varphi_j\}$, $\varphi_j \in D_{L^p}$ converges to a function $\varphi \in D_{L^p}$, in D_{L^p}, $(j \to \infty)$ if

$$\lim_{j\to\infty} \|D_t^\beta \varphi_j(t) - D_t^\beta \varphi(t)\|_{L^p} = 0$$

for every n-tuple of nonnegative integers β.
$D'_{L^p} = D'_{L^p}(\mathbf{R}^n)\,, \quad 1 < p \leq \infty$
is the space of all linear, continuous functionals on D_{L^q}, $\dfrac{1}{p} + \dfrac{1}{q} = 1$;

$D'_{L^1} = D'_{L^1}(\mathbf{R}^n)$ is the space of all linear, continuous functionals on $\dot{B}$, where $\dot{B}$ is the subspace of $B = D_{L^\infty}$ such that

$$\dot{B} = \{\varphi \in B \mid \lim_{|t|\to\infty} D^\beta \varphi(t) = 0, \text{for every } n\text{-tuple of nonnegative integer } \beta\}$$

Convergence in $\dot{B}$ is defined in the following way: a sequence $\{\varphi_j\}$, $\varphi_j \in \dot{B}$ converges to $\varphi \in \dot{B}$, in $\dot{B}$ $(j \to \infty)$ if

$$\lim_{j\to\infty} \|D_t^\beta \varphi_j(t) - D_t^\beta \varphi(t)\|_{L^\infty} = 0\,.$$

1. Main results

Theorem 1. *Let L be a subset of $H^p(\Pi^+)$, $1 \leq p < \infty$ and L' be the set of all regular distributions in D', generated with the boundary values of the functions of L. If L is bounded set in $H^p(\Pi^+)$, then L' is also bounded set in D'.*

Proof. Let $1 \leq p < \infty$ and L be a subset of $H^p(\Pi^+)$. With L^* we denote the set of the boundary values of the functions of L. It is clear, from the part 0.2, that $L^* \subset L^p(\mathbf{R})$. We will prove that L^* is bounded in $L^p(\mathbf{R})$.

Indeed, since L is bounded in $H^p(\Pi^+)$, there is a constant $\eta > 0$ such that $\|f\|_{H^p} \leq \eta$, for every $f \in L$.

Now, let $f^* \in L^*$ be arbitrary chosen. It means that f^* is the boundary value of some function f from the set L. Again, from the part 0.2, we have that $\|f^*\|_p = \|f\|_{H^p}$, and since $f \in L$, i.e. $\|f\|_{H^p} < \eta$, we have $\|f^*\|_p < \eta$. Because f^* was arbitrary chosen, we get that

$$L^* \quad \text{is bounded in} \quad L^p(\mathbf{R})\,. \tag{1}$$

It remains to prove the boundness of L' i.e. to find a constant M_B such that

$$\sup_{\varphi\in B} |\langle T_{f^*}, \varphi\rangle| \leq M_B\,,$$

for all $T_{f^*} \in L'$ and for all bounded subsets B of D.

Let B be any bounded subset of D.

Then:

$$\begin{aligned}&\text{there is a compact subset } \quad K \subset \mathbf{R}, \text{such that}\\ &\operatorname{supp}(\varphi) \subset K, \quad \text{for every} \quad \varphi \in B,\end{aligned} \tag{2}$$

and

$$\begin{aligned}&\text{for every} \quad \alpha \in \mathbf{N} \cup \{0\}, \ \text{there is a constant} \quad M_\alpha \ \text{such that}\\ &\sup_{t \in K} |D^\alpha \varphi(t)| \le M_\alpha\,, \quad \text{for every} \quad \varphi \in B\,.\end{aligned} \tag{3}$$

Now, let $\varphi \in B$, $T_{f^*} \in L'$ be arbitrary chosen. Then:

$$\begin{aligned}|\langle T_{f^*}, \varphi \rangle| &= \left| \int_{-\infty}^{\infty} f^*(x)\varphi(x)dx \right| \le \\ &\le \int_{-\infty}^{\infty} |f^*(x)|\,|\varphi(x)|dx \overset{(2)}{=} \\ &= \int_K |f^*(x)|\,|\varphi(x)|dx \overset{(3)}{\le} M_0 \int_K |f^*(x)|dx \le \\ &\le M_0 \left(\int_K |f^*(x)|^p dx \right)^{1/p} \left(\int_K |X_K(x)|^q dx \right)^{1/q} = \\ &= M_0 \|f^*\|_p \big(m(K)\big)^{1/q} \overset{(1)}{\le} M_0 \eta \big(m(K)\big)^{1/q} = M_B\,,\end{aligned}$$

where

$$X_K(x) = \begin{cases} 1, & x \in K \\ 0, & x \notin K \end{cases}$$

$m(K)$ is the Lebesque measure of K and q is such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Clearly, M_B depends of B.

So we proved that L' is bounded in D'.

It remains to examine if the converse is true i.e. whether the boundness of L' in D' implies the boundness of L in $H^p(\Pi^+)$, $1 \leq p < \infty$.

Theorem 1 implies some other results that are related with regularizations of distributions and convergence of sequence of distributions. For that matter, we will shortly mention the notion of convolution of functions, regularization of distributions and some known results which will be used. (all of them can be found in [7]).

Let f and g be two locally integrable functions on $\mathbf{R}^n$ such that one of them has a compact support. Their convolution is a functions $h = f * g$, defined with

$$h(x) = (f * g)(x) = \int\limits_{\mathbf{R}^n} f(x-y)g(y)\,dy = \int\limits_{\mathbf{R}^n} f(y)g(x-y)\,dy\,.$$

Let T, $U \in D'$ such that one of them, let's say U, has a compact support. Their convolution $T * U$ is defined with

$$\begin{aligned}\langle T * U, \varphi\rangle &= \langle T_x, \langle U_y,\, \varphi(x+y)\rangle\rangle = \\ &= \langle U_y,\, \langle T_x,\, \varphi(x+y)\rangle\rangle \quad \varphi \in D\end{aligned}$$

and $T * U \in D'$.

If instead of U, we consider a function g of D, then $T * g$ is infinitely differentiable function on $\mathbf{R}^n$, i.e. if $T \in D'$, $g \in D$ then $T * g$ is defined with

$$(T * g)(t) = \langle T_x,\, g(t-x)\rangle \quad \text{and} \quad (T * g)(t) \in C^\infty(\mathbf{R}^n)\,.$$

This convolution, $T * g$, is called regularization of T. The following theorem characterizes the bounded subsets of the distributions in D'.

Theorem . *A subset B' of distributions $T \in D'$ is bounded in D' if and only if, for every $\alpha \in D$, the regularization functions $T * \alpha$ are bounded on compact subsets of $\mathbf{R}^n$, when $T \in B'$.*

In theorem 1 we considered the set L' of regular distributions in D', generated with the boundary values of functions of L, $L \subset H^p(\Pi^+)$. So, a distribution T of L' is of the form

$$\langle T, \varphi\rangle = \int\limits_{\mathbf{R}} f^*(x)\varphi(x)\,dx\,, \quad \varphi \in D\,,$$

where $f^*(x)$ is the nontangential limit of $f(z) \in H^p$, and $f^*(x) \in L^p(\mathbf{R})$.

The regularization of such distribution with $\alpha \in D$ is

$$(T * \alpha)(t) = \langle T_x, \alpha(t-x) \rangle = \\ = \int_{\mathbf{R}} f^*(x)\alpha(t-x)\,dx = (f^* * \alpha)(t).$$

Now, from Theorem 1, the theorem that characterizes the bounded sets in D' and the above discussion, we have the following result:

Corollary 1. Let L be a bounded subset of $H^p(\Pi^+)$, $1 \le p < \infty$, and L^* be the set of the boundary values of the functions of L. Then, for every infinitely differentiable function, α, with compact support, the convolution functions $(f^* * \alpha)(t)$ are bounded on compact subsets of R, when $f^* \in L^*$.

Now, we will characterize the convergent sequences in the set of regular distributions generated with the boundary values of functions of H^p, $1 \le p < \infty$.

The following theorem characterizes the convergent sequences in D'.

Theorem. A sequence of distributions T_j converges to 0 in D' $(j \to \infty)$ if and only if, for every $\alpha \in D$, the regularization functions $T_j * \alpha$ converge uniformly to 0, on compact subsets of $\mathbf{R}^n$ $(j \to \infty)$.

From this theorem and the previous discussion, we have that the following result holds, i.e.

Corollary 2. *A* sequence of regular distributions $\{T_{f_j^*}\}$, generated with boundary values $f_j^*(t)$ of functions $f_j(z)$, of $H^p(\Pi^+)$, $1 \le p < \infty$, converges to 0 in D', $(j \to \infty)$ if and only if, for every infinitely differentiable function, α, on $\mathbf{R}$ with compact support, the convolution functions, $f_j^* * \alpha$, converge uniformly to 0, on compact subsets of $\mathbf{R}$ $(j \to \infty)$.

Now, let $\varphi(t) \in D$ be arbitrary chosen function of D and let $f^*(t)$ be the boundary value of a function $f(z) \in H^p(\Pi^+)$, $1 \le p < \infty$. From the part 0.2. we have that $f^*(t) \in L^p(\mathbf{R})$. Since $D \subset L^1$, we have that $\varphi(t) \in L^1(\mathbf{R})$. The theorem of Young (see [4]) claims that the convolution $f^* * \varphi$ belongs to L^p and even more

$$\|f^* * \varphi\|_p \le \|f^*\|_p \, \|\varphi\|_1 \,.$$

It is known the following characterization theorem of distributions in D'_{L^p} (see [7]).

Theorem. A distribution T belongs to D'_{L^p}, $1 \leq p < \infty$ if and only if, for every $\alpha \in D$, the regularization function $(T * \alpha)(t)$ belongs to L^p.

From this theorem and the above discussion, we have that the regular distributions T_{f^*}, generated with boundary functions f^*, of functions f, of $H^p(\Pi^+)$, $1 \leq p < \infty$ are distributions of D'_{L^p}.

In [1] it is proved the following result.

Theorem . Let $\{T_j\}$ be a sequence in D'_{L^p}, $p \in [1, \infty]$. It converges to 0 in D'_{L^p}, $j \to \infty$ if and only if, for every $\varphi \in D$, $T_j * \varphi$ converges to 0 in L^p, $(j \to \infty)$. Now, it is easy to see that the following result holds i.e.

Corollary 3. A sequence of regular distributions, $\{T_{f_j^*}\}$ generated with boundary values $f_j^*(t)$ of functions $f_j(z) \in H^p(\Pi^+)$, $1 \leq p < \infty$ converges to 0 in D'_{L^p} $(j \to \infty)$ if and only if for every $\alpha \in D$, the sequence of the convolution functions $f_j^* * \alpha$, converges to 0 in L^p, $(j \to \infty)$.

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ОГРАНИЧЕНИ ПОДМНОЖЕСТВА ОД ДИСТРИБУЦИИ ВО D' ГЕНЕРИРАНИ СО ГРАНИЧНИ ВРЕДНОСТИ НА ФУНКЦИИ ОД ПРОСТОРОТ H^p, $1 \le p < \infty$

Весна Манова-Ераковиќ

Апстракт: Во оваа работа е докажано дека ако подмножеството L од $H^p(\Pi^+)$, $1 \le p < \infty$ е ограничено во $H^p(\Pi^+)$, тогаш множеството од регуларни дистрибуции во D' генерирани со гранични вредности на функции од L е исто така ограничено во D'.

Исто така, дадени се и некои последици коишто следуваат од главниот резултат.

клучни зборови: дистрибуција, ограничено множество, простор H^p, $1 \le p < \infty$, гранична вредност на функција.

Природно-математички факултет
Институт за математика
П.Фах 162, 1000 Скопје
Македонија