FUZZY PAIRWISE STRONG PRECONTINUITY

Biljana Krsteska

Abstract. The concept of fuzzy strongly preopen sets has been introduced in fuzzy bitopological spaces. Some of their basic properties has been investigated and studied. The fuzzy strongly precontinuous, fuzzy strongly preopen and fuzzy strongly preclosed mappings have been generalized in fuzzy bitopological spaces. Some of their basic properties and relationships with other types of weaker forms of fuzzy pairwise continuous mappings has been investigated and studied.

Keywords and phrases: fuzzy topological space, fuzzy bitopological space, fuzzy strongly preopen set, fuzzy strongly preclosed set, fuzzy pairwise strong preconinuity, fuzzy pairwise strongly preopen mapping, fuzzy pairwise strongly preclosed mapping.

1. Introduction

The concept of fuzzy set was introduced by Zadeh in his classic paper [7]. Using the concept of fuzzy sets Chang [2] introduced the fuzzy topological spaces. Kandil [3] introduced and studied the notion of fuzzy bitopological spaces as a natural generalization of fuzzy topological spaces. Recently, Kumar [5,6] defined the $(\tau_i,\tau_j)-$fuzzy semiopen (semiclosed) sets, $(\tau_i,\tau_j)-$fuzzy preopen (preclosed) sets, $(\tau_i,\tau_j)-$fuzzy strongly semiopen sets, fuzzy pairwise semicontinuous mappings, fuzzy pairwise semiopen (semiclosed) mappings, fuzzy pariwise precontinuous mappings, fuzzy pairwise preopen (preclosed) mappings, fuzzy pairwise $\alpha-$continuous mappings and fuzzy pairwise $\alpha-$open (closed) mappings.

In this paper, we define the concept of $(\tau_i,\tau_j)-$fuzzy strongly preopen (preclosed) sets. We will prove that every $(\tau_i,\tau_j)-$fuzzy strongly preopen set is a $(\tau_i,\tau_j)-$fuzzy preopen set, but not conversely. Also, every $(\tau_i,\tau_j)-$fuzzy strongly semiopen set is a $(\tau_i,\tau_j)-$fuzzy strongly preopen set, but a $(\tau_i,\tau_j)-$fuzzy strongly preopen set may be not a $(\tau_i,\tau_j)-$fuzzy strongly semiopen set. Also, we will define the concept of fuzzy pairwise strong precontinuous mappings and fuzzy pairwise strongly preopen (preclosed) mappings. We will establish their properties and relationships with other classes of early defined weaker forms of fuzzy pairwise continuous mappings.

2. Preliminaries

Some notions and results that will be needed in this paper are recalled here.

A triple (X,τ_1,τ_2) consisting of a nonempty set X with two fuzzy topologies τ_1 and τ_2 on X is called a fuzzy bitopological spaces, shortly fbts. Throughout this paper, the indices i and j take values in $\{1,2\}$ and $i \neq j$. For a fuzzy set A of an fbts (X,τ_1,τ_2), $\tau_i-\operatorname{int}A$ and τ_j-clA means, the interior and closure of A with respect to the fuzzy topologies τ_i and τ_j.

Definition 2.1. [5,6] *Let A be a fuzzy set of an fbts (X,τ_1,τ_2). Then A is called*

(1) a $(\tau_i,\tau_j)-$fuzzy semiopen set if and only if there exists τ_i-fuzzy open set U such that $U \leq A \leq \tau_j-clU$.

(2) a $(\tau_i,\tau_j)-$fuzzy semiclosed set if and only if there exists τ_i-fuzzy closed set U such that $\tau_j-\operatorname{int}U \leq A \leq U$;

(3) a $(\tau_i,\tau_j)-$fuzzy preopen set if and only if $A \leq \tau_i-\operatorname{int}(\tau_j-clA)$.

(4) a $(\tau_i,\tau_j)-$fuzzy preclosed set if and only if $A \geq \tau_i-cl(\tau_j-\operatorname{int}A)$.

(5) a $(\tau_i,\tau_j)-$fuzzy strongly semiopen set if and only if

$$A \leq \tau_i-\operatorname{int}(\tau_j-cl(\tau_i-\operatorname{int}A)).$$

(6) a $(\tau_i,\tau_j)-$fuzzy strongly semiclosed set if and only if

$$A \geq \tau_i-cl(\tau_j-\operatorname{int}(\tau_i-clA)).$$

Definition 2.2. *Let A be a fuzzy set of an fbts (X,τ_1,τ_2).*

(1) The union of all $(\tau_i,\tau_j)-$fuzzy preopen set containing in A is called a $(\tau_i,\tau_j)-$fuzzy preinterior of A, denoted by $(\tau_i,\tau_j)-p\operatorname{int}A$.

(2) The intersection of all $(\tau_i,\tau_j)-$fuzzy preclosed set containing A is called a $(\tau_i,\tau_j)-$fuzzy strong preclosure of A, denoted by $(\tau_i,\tau_j)-pclA$..

Definition 2.3. [5,6] *A mapping $f:(X,\tau_1,\tau_2)\to(Y,\eta_1,\eta_2)$ from an fbts X into an fbts Y is called*

(1) a fuzzy pairwise semicontinuous if $f^{-1}(B)$ is a $(\tau_i,\tau_j)-$fuzzy semiopen set of X, for each η_i-fuzzy open set B of Y;

(2) a fuzzy pairwise precontinuous if $f^{-1}(B)$ is a $(\tau_i,\tau_j)-$fuzzy preopen set of X, for each η_i-fuzzy open set B of Y;

(3) a fuzzy pairwise strongly semicontinuous if $f^{-1}(B)$ is a $(\tau_i,\tau_j)-$fuzzy strongly semiopen set of X, for each η_i-fuzzy open set B of Y.

Definition 2.4. [5,6] *A mapping $f:(X,\tau_1,\tau_2)\to(Y,\eta_1,\eta_2)$ from an fbts X into an fbts Y is called*

(1) a fuzzy pairwise semiopen (semiclosed) if $f(A)$ is a $(\eta_i,\eta_j)-$fuzzy semiopen (semiclosed) set of Y, for each τ_i-fuzzy open (closed) set A of X;

(2) a fuzzy pairwise preopen (preclosed) if $f(A)$ *is a* $(\eta_i,\eta_j)-$*fuzzy preopen (preclosed) set of* Y, *for each* τ_i-*fuzzy open (closed) set* A *of* X;

(3) a fuzzy pairwise strongly semiopen (semiclosed) if $f(A)$ *is a* $(\eta_i,\eta_j)-$*fuzzy strongly semiopen (semiclosed) set of Y, for each* τ_i-*fuzzy open (closed) set* A *of* X.

For the definitions and results there are not explained in this paper we refer to the [1,4].

3.1. Fuzzy strongly preopen sets and fuzzy strongly preclosed set

Definition 3.1. *A fuzzy set* A *of an fbts* (X,τ_1,τ_2) *is called*

(1) a $(\tau_i,\tau_j)-$ *fuzzy strongly preopen set if and only if*

$$A\leq\tau_i-\operatorname{int}((\tau_j,\tau_i)-pclA);$$

(2) a $(\tau_i,\tau_j)-$*fuzzy strongly preclosed set if and only if*

$$A\geq\tau_i-cl((\tau_j,\tau_i)-p\operatorname{int}A).$$

Theorem 3.1. *A fuzzy set* A *of an fbts* (X,τ_1,τ_2) *is a* $(\tau_i,\tau_j)-$*fuzzy strongly preopen set if and only if* A^c *is a* $(\tau_i,\tau_j)-$ *fuzzy strongly preclosed set.*

Proof. Let A be any $(\tau_i,\tau_j)-$fuzzy strongly preopen set. It follows that $A\leq\tau_i-\operatorname{int}((\tau_j,\tau_i)-pclA)$. Then $A^c\geq\tau_i-cl((\tau_j,\tau_i)-p\operatorname{int}A)$, so A^c is a $(\tau_i,\tau_j)-$ fuzzy strongly preclosed set.

Conversely, let A^c be any $(\tau_i,\tau_j)-$fuzzy strongly preclosed set. Therefore $A^c\geq\tau_i-cl((\tau_j,\tau_i)-p\operatorname{int}A^c)$. Then $A^c\geq\tau_i-cl((\tau_j,\tau_i)-p\operatorname{int}A)$, so A^c is a $(\tau_i,\tau_j)-$fuzzy strongly preopen set.■

Theorem 3.2. *Let* A *be a fuzzy set of an fbts* (X,τ_1,τ_2).

(1) If A *is a* τ_i-*fuzzy open set, then* A *is a* $(\tau_i,\tau_j)-$*fuzzy strongly preopen set.*

(2) If A *is a* $(\tau_i,\tau_j)-$*fuzzy strongly semiopen set, then* A *is a* $(\tau_i,\tau_j)-$*fuzzy strongly preopen set.*

(3) If A *is a* $(\tau_i,\tau_j)-$*fuzzy strongly preopen set, then* A *is a* $(\tau_i,\tau_j)-$*fuzzy preopen set.*

Proof. (1) Let A be any τ_i-fuzzy open set. From $A\leq(\tau_j,\tau_i)-pclA$ it follows that $A=\tau_i-\operatorname{int}A\leq\tau_i-\operatorname{int}((\tau_j,\tau_i)-pclA)$, so A is a $(\tau_i,\tau_j)-$fuzzy strongly preopen set.

(2) Let A be any $(\tau_i,\tau_j)-$fuzzy strongly semiopen set. Since $(\tau_j,\tau_i)-pclA$ is a $(\tau_j,\tau_i)-$fuzzy preclosed set, we have

$A \leq \tau_i - \operatorname{int}(\tau_j - cl(\tau_i - \operatorname{int} A)) \leq \tau_i - \operatorname{int}(\tau_j - cl(\tau_i - \operatorname{int}((\tau_j, \tau_i) - pclA))) \leq$
$\leq \tau_i - \operatorname{int}((\tau_j, \tau_i) - pclA)),$

so A is a $(\tau_i, \tau_j)-$ fuzzy strongly preopen set.

(3) Let A be any $(\tau_i, \tau_j)-$ fuzzy strongly preopen set. From

$$A \leq \tau_i - \operatorname{int}((\tau_j, \tau_i) - pclA) \leq \tau_i - \operatorname{int}(\tau_j - clA)$$

it follows that A is a $(\tau_i, \tau_j)-$ fuzzy preopen set.■

Example 3.1. Let $X = \{a, b, c\}$ and A, B, C are fuzzy sets of X defined by

$A(a) = 0,3;$	$A(b) = 0,2;$	$A(c) = 0,2;$
$B(a) = 0,8;$	$B(b) = 0,8;$	$B(c) = 0,2;$
$C(a) = 0,8;$	$C(b) = 0,7;$	$C(c) = 0,6.$

We consider the fbts (X, τ, τ) where $\tau = \{0, A, B, A \wedge B, A \vee B, 1\}$. The fuzzy set C is $(\tau, \tau)-$ fuzzy strongly preopen set in fbts but C is not a $(\tau, \tau)-$ fuzzy strongly semiopen set. If we choose $\tau_1 = \{0, B, 1\}$, then the fuzzy set A is $(\tau_1, \tau_1)-$ fuzzy preopen set of fbts (X, τ_1, τ_1), which is not $(\tau_1, \tau_1)-$ fuzzy strongly preopen. ♦

In general, from the above example we can conclude that a $(\tau_i, \tau_j)-$ fuzzy preopen set of an fbts (X, τ_1, τ_2) may be not a $(\tau_i, \tau_j)-$ fuzzy strongly preopen set, and a $(\tau_i, \tau_j)-$ fuzzy strongly preopen set may be not a $(\tau_i, \tau_j)-$ fuzzy strongly semiopen set.

Theorem 3.3. *(1) Any union of* $(\tau_i, \tau_j)-$*fuzzy strongly preopen set is a* $(\tau_i, \tau_j)-$*fuzzy strongly preopen set.*

(2) Any intersection of $(\tau_i, \tau_j)-$*fuzzy strongly preclosed set is a* $(\tau_i, \tau_j)-$ *fuzzy strongly preclosed set.*

Proof. (1) Let $\{A_\alpha\}_{\alpha \in I}$ be any collection of $(\tau_i, \tau_j)-$ fuzzy strongly preopen sets. For each $\alpha \in I$ we have $A_\alpha \leq \tau_i - \operatorname{int}((\tau_j, \tau_i) - pclA_\alpha)$. Then

$$\vee_{\alpha \in I} A_\alpha \leq \vee_{\alpha \in I} \tau_i - \operatorname{int}((\tau_j, \tau_i) - pclA_\alpha) \leq$$
$$\leq \tau_i - \operatorname{int}(\vee_{\alpha \in I}(\tau_j, \tau_i) - pclA_\alpha) \leq \tau_i - \operatorname{int}((\tau_j, \tau_i) - pcl(\vee_{\alpha \in I} A_\alpha))$$

so $\vee_{\alpha \in I} A_\alpha$ is a $(\tau_i, \tau_j)-$ fuzzy strongly preopen set.

(2) Let $\{A_\alpha\}_{\alpha \in I}$ be any collection of $(\tau_i, \tau_j)-$ fuzzy strongly preclosed sets. Thus $\{A_\alpha{}^c\}_{\alpha \in I}$ is a family of a $(\tau_i, \tau_j)-$ fuzzy strongly preopen set. From (1) it follows that $\vee_{\alpha \in I} A_\alpha{}^c$ is a $(\tau_i, \tau_j)-$ fuzzy strongly preopen set. From the relation $(\vee_{\alpha \in I} A_\alpha{}^c)^c = \wedge_{\alpha \in I} A_\alpha$ we obtain the conclusions.■

As a consequence of the previous theorem we have that the triple $(X,FSP(\tau_1),FSP(\tau_2))$ is a fuzzy supra-bitopological spaces. In the next examples we will show that it's not a fuzzy bitopological spaces. More precisely, the intersection of two $(\tau_i,\tau_j)-$fuzzy strongly preopen set may not be a fuzzy strongly preopen set. Similarly, the union of two $(\tau_i,\tau_j)-$fuzzy strongly preclosed set may not be a fuzzy strongly preclosed set.

Example 3.2. We can verify the above conclusion if we consider the fbts (X,τ,τ) in the Example 3.1. The fuzzy set D defined as

$$D(a)=0{,}4; \qquad D(b)=0{,}2; \qquad C(c)=0{,}8;$$

is a $(\tau,\tau)-$fuzzy strongly preopen set, but $B\wedge D$ is not a $(\tau,\tau)-$fuzzy strongly preopen set. Also, B^c and D^c is a $(\tau,\tau)-$fuzzy strongly preclosed set, but $B^c\vee D^c$ is not a $(\tau,\tau)-$fuzzy strongly preclosed set.♦

Definition 3.2. *Let A be a fuzzy set of an fbts (X,τ_1,τ_2).*

(1) The union of all $(\tau_i,\tau_j)-$fuzzy strongly preopen set containing in A is called a $(\tau_i,\tau_j)-$fuzzy strong preinterior of A, denoted by $(\tau_i,\tau_j)-sp\operatorname{int}A$.

(2) The intersection of all $(\tau_i,\tau_j)-$fuzzy strongly preclosed set containing A is called a $(\tau_i,\tau_j)-$fuzzy strong preinterior of A, denoted by $(\tau_i,\tau_j)-spclA$.

Theorem 3.4. *Let A be a fuzzy set of an fbts (X,τ_1,τ_2). Then*

(1) A is a fuzzy strongly preopen if and only if $A=(\tau_i,\tau_j)-sp\operatorname{int}A$.

(2) A is a fuzzy strongly preclosed if and only if $A=(\tau_i,\tau_j)-spclA$.

Proof. Its follows from the Definition 3.2 and Theorem 3.3.■

Theorem 3.5. *Let A and B be a fuzzy sets of an fbts (X,τ_1,τ_2). Then,*

(1) $\tau_i-\operatorname{int}A\le(\tau_i,\tau_j)-s\sin tA\le(\tau_i,\tau_j)-sp\operatorname{int}A\le(\tau_i,\tau_j)-p\operatorname{int}A\le A\le$

$\le(\tau_i,\tau_j)-pclA\le(\tau_i,\tau_j)-spclA\le(\tau_i,\tau_j)-ssclA\le\tau_i-clA.$

(2) If $A\le B$, then $(\tau_i,\tau_j)-sp\operatorname{int}A\le(\tau_i,\tau_j)-sp\operatorname{int}B$ and

$(\tau_i,\tau_j)-spclA\le(\tau_i,\tau_j)-spclB.$

(3) $(\tau_i,\tau_j)-sp\operatorname{int}((\tau_i,\tau_j)-sp\operatorname{int}A)=(\tau_i,\tau_j)-sp\operatorname{int}A$ *and*

$(\tau_i,\tau_j)-spcl((\tau_i,\tau_j)-spclA)=(\tau_i,\tau_j)-spclA$

Proof. It can be proven by using the Definition 3.2 and Theorem 3.4.■

Theorem 3.6. *Let A be a fuzzy set of an fbts (X,τ_1,τ_2). Then the following statement holds:*

(1) $(\tau_i,\tau_j)-spclA^c=((\tau_i,\tau_j)-sp\operatorname{int}A)^c.$

(2) $(\tau_i,\tau_j)-sp\operatorname{int}A^c=((\tau_i,\tau_j)-spclA)^c.$

Proof.

(1) $((\tau_i,\tau_j)-sp\operatorname{int}A)^c = (\vee\{B \mid B \leq A,\ B \in (\tau_i,\tau_j)-FSPO\})^c =$
$= \wedge\{B^c \mid B \leq A,\ B \in (\tau_i,\tau_j)-FSPO\}) =$
$= \wedge\{D \mid D \geq A^c, D \in (\tau_i,\tau_j)-FSPC\}) = (\tau_i,\tau_j)-spclA^c.$

(2) $((\tau_i,\tau_j)-spclA)^c = (\wedge\{B \mid B \geq A,\ B \in (\tau_i,\tau_j)-FSPC\})^c =$
$= \vee\{B^c \mid B \geq A,\ B \in (\tau_i,\tau_j)-FSPC\} =$
$= \vee\{D \mid D \leq A^c, D \in (\tau_i,\tau_j)-FSP0\}) = (\tau_i,\tau_j)-sp\operatorname{int}A^c.$ ■

Theorem 3.7. *Let* (X,τ_1,τ_2) *and* (Y,σ_1,σ_2) *be fbt's such that* X *is product related to* Y. *Then the product* $A \times B$ *of a* $(\tau_i,\tau_j)-$*fuzzy strongly preopen set* A *of* X *and a* $(\sigma_i,\sigma_j)-$*fuzzy strongly preopen set* B *of* Y *is a* $(\delta_1,\delta_j)-$*fuzzy preopen set of the product spaces* $(X \times Y,\rho_1,\rho_2)$, *where* ρ_k, $k = 1,2$ *is the product topology generated by* τ_k *and* σ_k.

Proof. Since A is a $(\tau_i,\tau_j)-$ fuzzy strongly preopen set and B is a $(\sigma_i,\sigma_j)-$ fuzzy strongly preopen set, $A \leq \tau_i - \operatorname{int}((\tau_j,\tau_i)-pclA)$ and $B \leq \sigma_i - \operatorname{int}((\sigma_j,\sigma_i)-pclB)$, then by Theorem 3.10 [1] and Theorem 3.10 [4] we have

$$A \times B \leq \tau_i - \operatorname{int}((\tau_j,\tau_i)-pclA) \times \sigma_i - \operatorname{int}((\sigma_j,\sigma_i)-pclB) =$$
$$= \rho_i - \operatorname{int}((\tau_j,\tau_i)-pclA \times ((\sigma_j,\sigma_i)-pclB) \leq$$
$$\leq \rho_i - \operatorname{int}(\tau_j - clA \times \sigma_j - clB) = \rho_i - \operatorname{int}(\rho_j - cl(A \times B)).$$ ■

4. Fuzzy pairwise strong precontinuity

Definition 4.1. *A mapping* $f:(X,\tau_1,\tau_2) \to (Y,\eta_1,\eta_2)$ *from an fbts* X *into an fbts* Y *is called a fuzzy pairwise strong precontinuous if* $f^{-1}(B)$ *is a* $(\tau_i,\tau_j)-$*fuzzy strongly preopen set of* Y *for each* η_i-*fuzzy open set* B *of* Y.

Remark 4.1. From the above definitions it is not difficult to conclude that the following diagram of implications is true.

fuzzy pairwise continuity ⇒ fuzzy pairwise strong semicontinuity ⇒ fuzzy pairwise semicontinuity

⇓ ⇓

fuzzy parwise strong precontinuity

⇓

fuzzy pairwise precontinuity

Obviously, the converse does not hold as is shown by the following examples.

Example 4.1. Let $X = \{a,b,c\}$ and A,B,C are fuzzy sets of X defined by

$A(a) = 0,3;$ $A(b) = 0,2;$ $A(c) = 0,7;$
$B(a) = 0,8;$ $B(b) = 0,8;$ $B(c) = 0,4;$

$C(a)=0{,}8;\qquad C(b)=0{,}7;\qquad C(c)=0{,}6.$

We consider the fbts's (X,τ_1,τ_1), (X,τ_2,τ_2) and (X,τ_3,τ_3) where $\tau_1=\{0,A,B,A\wedge B,A\vee B,1\}$, $\tau_2=\{0,A,1\}$ and $\tau_3=\{0,C,1\}$. Then $f=id:(X,\tau_1,\tau_1)\to(X,\tau_2,\tau_2)$ is fuzzy pairwise precontinuous mapping but is not a fuzzy pairwise strong precontinuous mapping. Similarly, $f=id:(X,\tau_1,\tau_1)\to(X,\tau_3,\tau_3)$ is fuzzy pairwise strong precontinuous mapping but is neither fuzzy pairwise strong semicontinuous mapping nor fuzzy pairwise continuous mapping.♦

Theorem 4.1. *Let* $f:(X,\tau_1,\tau_2)\to(Y,\eta_1,\eta_2)$ *be a mapping from an fbts* (X,τ_1,τ_2) *to an fbts* (Y,η_1,η_2). *Then the following statements are equivalent:*

(i) f *is a fuzzy pairwise strong precontinuos mapping;*

(ii) $f^{-1}(B)$ *is a* $(\tau_i,\tau_j)-$*fuzzy strongly preclosed set of fbts* X, *for each* η_i- *fuzzy closed set of fbts* Y;

(iii) $f((\tau_i,\tau_j)-spclA)\leq\eta_i-clf(A)$, *for each fuzzy set* A *of fbts* X;

(iv) $(\tau_i,\tau_j)-spclf^{-1}(B)\leq f^{-1}(\eta_i-clB)$, *for each fuzzy set* B *of fbts* Y;

(v) $f^{-1}(\eta_i-\operatorname{int}B)\leq(\tau_i,\tau_j)-sp\operatorname{int}f^{-1}(B)$, *for each fuzzy set* B *of fbts* Y.

Proof. $(i)\Rightarrow(ii)$ Let B be a η_i- fuzzy closed set of fbts Y. Then B^c is a η_i- fuzzy open set of fbts Y. By assumption and Definition 4.1 we obtain that $f^{-1}(B^c)$ is a $(\tau_i,\tau_j)-$ fuzzy strongly preopen set of fbts X. From $f^{-1}(B^c)=(f^{-1}(B))^c$ it follows that $f^{-1}(B)$ is a $(\tau_i,\tau_j)-$ fuzzy strongly preclosed set of fbts X.

$(ii)\Rightarrow(iii)$ Let A be any fuzzy set of fbts X. Then $\eta_i-clf(A)$ is η_i- fuzzy closed set of fbts Y, so $f^{-1}(\eta_i-clf(A))$ is a $(\tau_i,\tau_j)-$ fuzzy strongly preclosed set of X. From

$$(\tau_i,\tau_j)-spclA\leq(\tau_i,\tau_j)-spclf^{-1}(f(A))\leq(\tau_i,\tau_j)-spclf^{-1}(\eta_i-clf(A))=$$
$$=f^{-1}(\eta_i-clf(A))$$

it follows that $f((\tau_i,\tau_j)-spclA)\leq\eta_i-clf(A)$.

$(iii)\Rightarrow(iv)$ Let B be any fuzzy set of fbts Y. According to the assumption it follows that $f((\tau_i,\tau_j)-spclf^{-1}(B))\leq\eta_i-clf(f^{-1}(B))$. Then

$$(\tau_i,\tau_j)-spclf^{-1}(B)\leq f^{-1}f((\tau_i,\tau_j)-spclf^{-1}(B))\leq f^{-1}(\eta_i-clB).$$

$(iv)\Rightarrow(v)$ It can be proved by using the complement.

$(v)\Rightarrow(i)$ Let B be a η_i- fuzzy open set of fbts Y. According to the assumption we have

$$f^{-1}(B)=f^{-1}(\eta_i-\operatorname{int}B)\leq(\tau_i,\tau_j)-sp\operatorname{int}f^{-1}(B).$$

Therefore $f^{-1}(B)=(\tau_i,\tau_j)-sp\text{int}f^{-1}(B)$, so $f^{-1}(B)$ is a $(\tau_i,\tau_j)-$ fuzzy strongly preopen set of fbts *X*. Hence f is fuzzy pairwise strong precontinuous mapping.∎

Theorem 4.2. *Let* $f:(X,\tau_1,\tau_2)\to(Y,\eta_1,\eta_2)$ *be a mapping from an fbts* (X,τ_1,τ_2) *to an fbts* (Y,η_1,η_2). *The following statements are equivalent:*

(i) f *is a fuzzy pairwise strong precontinuous mapping;*

(ii) $f^{-1}(\eta_i-\text{int}B)\le\tau_i-\text{int}((\tau_j,\tau_i)-pclf^{-1}(B))$, *for each fuzzy set* B *of fbts* Y;

(iii) $\tau_i-cl((\tau_j,\tau_i)-p\text{int}f^{-1}(B))\le f^{-1}(\eta_i-clB)$, *for each fuzzy set* B *of fbts* Y;

(iv) $f(\tau_i-cl((\tau_j,\tau_i)-p\text{int}A)\le\eta_i-clf(A)$, *for each fuzzy set* A *of fbts* X.

Proof. $(i)\Rightarrow(ii)$ Let B be any fuzzy set of fbts *Y*. According to the assumption $f^{-1}(\eta_i-\text{int}B)$ is a $(\tau_i,\tau_j)-$ fuzzy strongly preopen set of *X*. Therefore $f^{-1}(\eta_i-\text{int}B)\le\tau_i-\text{int}((\tau_j,\tau_i)-pclf^{-1}(\eta_i-\text{int}B))\le$

$$\le\tau_i-\text{int}((\tau_j,\tau_i)-pclf^{-1}(B)).$$

$(ii)\Rightarrow(iii)$ It can be proven by using the complement.

$(iii)\Rightarrow(iv)$ Let A be any fuzzy set of X. For $B=f(A)$ we have $A\le f^{-1}(B)$. According to the assumption we have

$$\tau_i-cl((\tau_j,\tau_i)-p\text{int}A)\le\tau_i-cl((\tau_j,\tau_i)-p\text{int}f^{-1}(B))\le f^{-1}(\eta_i-clB).$$

Hence $f(\tau_i-cl((\tau_j,\tau_i)-p\text{int}A)\le\eta_i-clB=\eta_i-clf(A)$.

$(iv)\Rightarrow(i)$ Let B be any η_i- fuzzy closed set of *Y*. According to the assumption we have

$$f(\tau_i-cl((\tau_j,\tau_i)-p\text{int}f^{-1}(B))\le\eta_i-clf(f^{-1}(B))\le\eta_i-clB=B.$$

Therefore $\tau_i-cl((\tau_j,\tau_i)-p\text{int}f^{-1}(B))\le$

$$\le f^{-1}f(\tau_i-cl((\tau_j,\tau_i)-p\text{int}f^{-1}(B))\le f^{-1}(B).$$ ∎

Theorem 4.3. *Let* $f:(X,\tau_1,\tau_2)\to(Y,\eta_1,\eta_2)$ *be a bijective mapping from an fbts* (X,τ_1,τ_2) *to an fbts* (Y,η_1,η_2). *The following statements are equivalent:*

(i) f *is a fuzzy pairwise strong precontinuous mapping;*

(ii) $\tau_i-\text{int}f(A)\le f((\tau_i,\tau_j)-sp\text{int}A)$, *for each fuzzy set* A *of fbts* X.

Proof. $(i)\Rightarrow(ii)$ Let A be any fuzzy set of fbts *X*. Then $f^{-1}(\eta_i-\text{int}f(A))$ is a $(\tau_i,\tau_j)-$ fuzzy strongly preopen set of fbts *X*. Since *f* is injective mapping, from the Theorem 4.1 (v) we have

$$f^{-1}(\eta_i-\text{int}f(A))\le(\tau_i,\tau_j)-sp\text{int}f^{-1}(f(A))=(\tau_i,\tau_j)-sp\text{int}A.$$

Since f is surjective mapping we have

$$\eta_i - \operatorname{int} f(A) = f(f^{-1}(\eta_i - \operatorname{int} f(A)) \leq f((\tau_i,\tau_j) - sp\operatorname{int} A).$$

$(ii) \Rightarrow (i)$ Let B be any $\eta_i -$ fuzzy open set of fbts Y. Then $B = \eta_i - \operatorname{int} B$. Since f is injective mapping, according to the assumption we have

$$f^{-1}(B) \leq f^{-1}f((\tau_i,\tau_j) - sp\operatorname{int} f^{-1}(B) = (\tau_i,\tau_j) - sp\operatorname{int} f^{-1}(B).$$

Hence $f^{-1}(B) = (\tau_i,\tau_j) - sp\operatorname{int} f^{-1}(B)$, so $f^{-1}(B)$ is a $(\tau_i,\tau_j)-$ fuzzy strongly preopen set of fbts X. ■

Theorem 4.4. *Let* (X_1,τ_1,τ_2), (X_2,δ_1,δ_2), (Y_1,ω_1,ω_2) *and* (Y_2,σ_1,σ_2) *be fts's such that* X_1 *is product related* X_2 *[1]. Then the product* $f_1 \times f_2 : (X_1 \times X_2, \alpha_1, \alpha_2) \to (Y_1 \times Y_2, \beta_1, \beta_2)$, *where* α_k *(resp.* β_k*) is the fuzzy product topology generated by* τ_k *and* δ_k *(resp.* ω_k *and* σ_k*) for* $k = 1,2$, *of fuzzy pairwise strong precontinuous mappings* $f_1 : (X_1,\tau_1,\tau_2) \to (Y_1,\omega_1,\omega_2)$ *and* $f_2 : (X_2,\delta_1,\delta_2) \to (Y_2,\sigma_1,\sigma_2)$ *is a fuzzy pairwise precontinuous mapping..*

Proof. Let $W = \vee_{m,n}(U_m \times V_n)$, where U_m's are $\omega_i -$ fuzzy open set of Y_1 and V_n's are $\sigma_i -$ fuzzy open sets of Y_2, be any $\beta_i -$ fuzzy open set of $Y_1 \times Y_2$. Then

$$(f_1 \times f_2)^{-1})(W) = \vee_{m,n}[(f_1 \times f_2)^{-1}(U_m \times V_n)] = \vee_{m,n}[f_1^{-1}(U_m) \times f_2^{-1}(V_n)].$$

Since f_1 and f_2 are fuzzy pairwise strong precontinuous mappings, $f_1^{-1}(U_m)$'s are $(\tau_i,\tau_j)-$ fuzzy strongly preopen sets of X_1 and $f_2^{-1}(V_n)$'s are $(\delta_i,\delta_j)-$ fuzzy strongly preopen sets of X_2. By Theorem 4.3 [4] and Theorem 4.7 [4] it follows that $(f_1 \times f_2)^{-1}(W)$ is a $(\alpha_i,\alpha_j)-$ fuzzy preopen set of fbts $X_1 \times X_2$. Hence $f_1 \times f_2$ is fuzzy pairwise precontinuous mapping.■

Theorem 4.5. *Let* $f : (X,\tau_1,\tau_2) \to (Y,\delta_1,\delta_2)$ *be a mapping from an fbts* X *to another fbts* Y. *Then if the graph* $g : (X,\tau_1,\tau_2) \to (X \times Y, \alpha_1, \alpha_2)$ *of* f, *defined by* $g(x) = (x, f(x))$ *for each* $x \in X$, *is the fuzzy strong precontinuous mapping, then* f *is also fuzzy strongly precontinuous mapping.*

Proof. Let V be any $\delta_i -$ fuzzy open set of fbts Y. Then by Lemma 2.4 of [1], we have $f^{-1}(V) = 1 \wedge f^{-1}(V) = g^{-1}(1 \times V)$.

Since g is fuzzy strong precontinuous and $1 \times V$ is a $\alpha_i -$ fuzzy open set of $X \times Y$, $f^{-1}(V)$ is a $(\tau_i,\tau_j)-$ fuzzy strongly preopen set of X.■

5. Fuzzy pairwise strongly preopen (preclosed) mappings

Definition 5.1. *A mapping* $f : (X,\tau_1,\tau_2) \to (Y,\eta_1,\eta_2)$ *from an fbts* X *into an fbts* Y *is called*

(1) a fuzzy pairwise strongly preopen if $f(A)$ is a $(\eta_i,\eta_j)-$fuzzy strongly preopen set of fbts Y for each τ_i-fuzzy open set A of fbts X;

(2) a fuzzy pairwise strongly preclosed if $f(A)$ is a $(\eta_i,\eta_j)-$fuzzy strongly preclosed set of fbts Y for each for each τ_i-fuzzy closed set A of fbts X.

Remark 5.1. From the above definitions it is not difficult to conclude that the following diagram of implications is true.

fuzzy pairwise open ⇒ fuzzy pairwise strongly semiopen ⇒ fuzzy pairwise semiopen
(closed) mapping (closed) mapping (semiclosed) mapping

⇓ ⇓

fuzzy parwise strongly preopen (preclosed) mapping

⇓

fuzzy pairwise preopen (preclosed) mapping

Obviously, the converse does not hold as is shown by the following examples.

Example 5.1. We consider the Example 4.1. The mapping $f=id:(X,\tau_3,\tau_3)\to(X,\tau_1,\tau_1)$ is a fuzzy parwise strongly preopen (preclosed) but it is not fuzzy pairwise strongly semiopen (semiclosed). Also, f is not fuzzy pairwise open (closed). Similarly the mapping $f=id:(X,\tau_2,\tau_2)\to(X,\tau_1,\tau_1)$ is a fuzzy pairwise preopen (preclosed), but it is not fuzzy pairwise strongly preopen (preclosed).♦

Theorem 5.1. *Let $f:(X,\tau_1,\tau_2)\to(Y,\eta_1,\eta_2)$ be a mapping from an fbts (X,τ_1,τ_2) to an fbts (Y,η_1,η_2). The following statements are equivalent:*

(i) f is a fuzzy strongly preopen mapping;

(ii) $f(\tau_i-\operatorname{int}A)\le(\eta_i,\eta_j)-sp\operatorname{int}f(A)$, for each fuzzy set A of fbts X;

(iii) $\tau_i-\operatorname{int}f^{-1}(B)\le f^{-1}((\eta_i,\eta_j)-sp\operatorname{int}B)$, for each fuzzy set B of fbts Y.

(iv) $f^{-1}((\eta_i,\eta_j)-spclB)\le\tau_i-clf^{-1}(B)$, for each fuzzy set B of fbts Y.

Proof. $(i)\Rightarrow(ii)$ Let A be any fuzzy set of fbts X. From the assumption it follows that $f(\tau_i-\operatorname{int}A)$ is a $(\eta_i,\eta_j)-$fuzzy strongly preopen set of fbts Y Therefore

$$f(\tau_i-\operatorname{int}A)=(\eta_i,n_j)-sp\operatorname{int}f(\tau_i-\operatorname{int}A)\le(\eta_i,n_j)-sp\operatorname{int}f(A)).$$

$(ii)\Rightarrow(iii)$ Let B be any fuzzy set of fbts Y. From the assumption it follows that $f(\tau_i-\operatorname{int}f^{-1}(B))\le(\eta_i,n_j)-sp\operatorname{int}f(f^{-1}(B))\le(\eta_i,n_j)-sp\operatorname{int}B$. Hence

$$\tau_i-\operatorname{int}f^{-1}(B)\le f^{-1}f(\tau_i-\operatorname{int}f^{-1}(B))\le f^{-1}((\eta_i,n_j)-sp\operatorname{int}B).$$

$(iii) \Rightarrow (iv)$ It can be proved by using the complement.

$(iv) \Rightarrow (i)$ Let A be any τ_i − fuzzy open set of fbts X. Then $\tau_i - \operatorname{int} A = A$. According to assumption we have

$$f^{-1}((\eta_i,\eta_j)-spclf(A)^c) \leq \tau_i - clf^{-1}(f(A)^c) = \tau_i - cl(f^{-1}f(A))^c \leq \tau_i - clA^c,$$

so $\tau_i - \operatorname{int} A \leq f^{-1}((\eta_i,\eta_j)-sp\operatorname{int} f(A))$.

From the last inclusion we obtain

$$f(A) \leq ff^{-1}((\eta_i,\eta_j)-sp\operatorname{int} f(A) \leq (\eta_i,\eta_j)-sp\operatorname{int} f(A),$$

so $f(A) = (\eta_i,\eta_j)-sp\operatorname{int} f(A)$.■

Theorem 5.2. *Let* $f:(X,\tau_1,\tau_2) \to (Y,\eta_1,\eta_2)$ *be a mapping from an fbts* (X,τ_1,τ_2) *to an fbts* (Y,η_1,η_2). *Then* f *is fuzzy strongly preclosed mapping if and only if* $(\eta_i,\eta_j)-spclf(A) \leq f(\tau_i - clA)$, *for each fuzzy set* A *of fbts* X.

Proof. Let f be a fuzzy strongly preclosed mapping and let A be any fuzzy set of fbts X. Then

$$f(\tau_i - clA) = (\eta_i,\eta_j)-spclf(\tau_i - clA) \geq (\eta_i,\eta_j)-spclf(A).$$

Conversely, let A be any fuzzy closed set of fbts X. From

$$f(A) = f(\tau_i - clA) \geq (\eta_i,\eta_j)-spclf(A),$$

we can conclude that f is a fuzzy strongly preclosed mapping.■

Theorem 5.3. *Let* $f:(X,\tau_1,\tau_2) \to (Y,\eta_1,\eta_2)$ *be a mapping from an fbts* (X,τ_1,τ_2) *to an fbts* (Y,η_1,η_2). *Then the following statements are true:*

(1) f *is a fuzzy strongly preopen mapping if and only if* $f(\tau_i - \operatorname{int} A) \leq \eta_i - \operatorname{int}((\eta_j,\eta_i)-pclf(A))$, *for each fuzzy set* A *of fbts* X.

(2) f *is a fuzzy strongly preclosed mapping if and only if* $\eta_i - cl((\eta_j,\eta_i)-p\operatorname{int} f(A)) \leq f(\tau_i - clA)$, *for each fuzzy set* A *of fbts* X.

Proof. (1) Let f be a fuzzy strongly preopen mapping and A be any set of X. Then $f(\eta_i - \operatorname{int} A)$ is a η_i − fuzzy strongly preopen set of Y, so

$$f(\tau_i - \operatorname{int} A) \leq \eta_i - \operatorname{int}((\eta_j,\eta_i)-pclf(\tau_i - \operatorname{int} A)) \leq \eta_i - \operatorname{int}((\eta_j,\eta_i)-pclf(A)).$$

(2) It can be proved in a similar manner as (1).■

Theorem 5.4. *Let* $f:(X,\tau_1,\tau_2) \to (Y,\eta_1,\eta_2)$ *be a bijective mapping from an fbts* (X,τ_1,τ_2) *to an fbts* (Y,η_1,η_2). *Then* f *is a fuzzy pairwise strongly preopen mapping if and only if* f *is a fuzzy pairwise strongly preclosed mapping.*

Proof. It can be proved by using complement and the relation $f(A^c) = f(A)^c$, for each fuzzy set A of X.■

Corollary 5.5. *Let* $f:(X,\tau_1,\tau_2) \to (Y,\eta_1,\eta_2)$ *be a bijective mapping from an fbts* (X,τ_1,τ_2) *to an fbts* (Y,η_1,η_2). (Y,η_1,η_2). *The following statements are equivalent:*

(i) f *is a fuzzy pairwise strongly preclosed mapping;*

(ii) $f^{-1}((\eta_i,\eta_j)-spclB)\leq\tau_i-clf^{-1}(B)$, *for each fuzzy set* B *of fbts* Y.

(iii) $\tau_i-\operatorname{int}f^{-1}(B)\leq f^{-1}((\eta_i,\eta_j)-sp\operatorname{int}B)$, *for each fuzzy set* B *of fbts* Y.

(iv) $f(\tau_i-\operatorname{int}A)\leq(\eta_i,\eta_j)-sp\operatorname{int}f(A)$, *for each fuzzy set* A *of fbts* X. ■

Corollary 5.6. *Let* $f:(X,\tau_1,\tau_2)\to(Y,\eta_1,\eta_2)$ *be a bijective mapping from an fbts* (X,τ_1,τ_2) *to an fbts* (Y,η_1,η_2). *Then* f *is fuzzy pairwise strongly preopen mapping if and only if* $(\eta_i,\eta_j)-spclf(A)\leq f(\tau_i-clA)$, *for each fuzzy set* A *of fbts* X. ■

Theorem 5.7. *Let* $f:(X,\tau_1,\tau_2)\to(Y,\eta_1,\eta_2)$ *be a mapping from an fbts* (X,τ_1,τ_2) *to an fbts* (Y,η_1,η_2). *Then* f *is fuzzy strongly preopen mapping if and only if for each fuzzy set* B *of fbts* Y *and each* τ_i-*fuzzy closed set* A *of* X, $f^{-1}(B)\leq A$, *there exists a* $(\eta_i,\eta_j)-$*fuzzy strongly preclosed set* C *of* Y *such that* $B\leq C$ *and* $f^{-1}(C)\leq A$.

Proof. Let B be any fuzzy set of Y and let A be any τ_i-fuzzy closed set of fbts X such that $f^{-1}(B)\leq A$. Then $A^c\leq f^{-1}(B^c)$, so $f(A^c)\leq ff^{-1}(B^c)\leq B^c$. Since A^c is a τ_i-fuzzy open set, $f(A^c)$ is a $(\eta_i,\eta_j)-$fuzzy strongly preclosed set, so $f(A^c)\leq(\eta_i,\eta_j)-sp\operatorname{int}B^c$. Hence $A^c\leq f^{-1}f(A^c)\leq f^{-1}((\eta_i,\eta_j)-sp\operatorname{int}B^c)$. Thus result follows for $C=(\eta_i,\eta_j)-spclB$.

Conversely, let U be any τ_i-fuzzy open set of X. We will show that $f(U)$ is a $(\eta_i,\eta_j)-$fuzzy strongly preopen set of Y. From $U\leq f^{-1}f(U)$ follows that $U^c\geq(f^{-1}f(U))^c\geq f^{-1}f(U)^c$ where U^c is τ_i-fuzzy closed set of X. Hence there is a $(\eta_i,\eta_j)-$fuzzy strongly precolsed B of Y such that $B\geq f(U)^c$ and $f^{-1}(B)\leq U^c$. From $B\geq f(U)^c$ follows that $B\geq(\eta_i,\eta_j)-spclf(U)^c$, so $B^c\leq(\eta_i,\eta_j)-(spclf(U)^c)^c\leq(\eta_i,\eta_j)-sp\operatorname{int}f(U)$. From $f^{-1}(B)\leq U^c$ we have $B\geq f^{-1}(B^c)\geq U$, so $B^c\geq ff^{-1}(B^c)\geq f(U)$.

Hence $f(U)=(\eta_i,\eta_j)-sp\operatorname{int}f(U)$. Thus $f(U)$ is a $(\eta_i,\eta_j)-$fuzzy strongly preopen set, so f is a fuzzy strongly preopen mapping.■

Theorem 5.8. *Let* $f:(X,\tau_1,\tau_2)\to(Y,\eta_1,\eta_2)$ *be a mapping from an fbts* (X,τ_1,τ_2) *to an fbts* (Y,η_1,η_2). *Then* f *is fuzzy strongly preclosed mapping if and only if for each fuzzy set* B *of fbts* Y *and each* τ_i-*fuzzy open set* A *of* X, $f^{-1}(B)\leq A$, *there exists a* $(\eta_i,\eta_j)-$*fuzzy strongly preopen set* C *of* Y *such that* $B\leq C$ *and* $f^{-1}(C)\leq A$.

Proof. It can be proved in a similar manner as Theorem 5.7. ■

Theorem 5.9. *Let $f: X \to Y$ and $g: Y \to Z$ be a mappings, where (X, τ_1, τ_2), (Y, η_1, η_2) and (Z, σ_1, σ_2) be fbts's. If f is a fuzzy pairwise open (closed) mapping and g is fuzzy pairwise strongly preopen (preclosed) then gf is fuzzy pairwise strongly preopen (preclosed) mapping.*

Proof. For any τ_i – fuzzy open set A of fbts X, we have $(gf)(A) = g(f(A))$. Since f is a fuzzy pairwise open (closed) mapping and g is fuzzy pairwise strongly preopen (preclosed) we obtain that $(gf)(A)$ is a (σ_i, σ_j) – fuzzy strongly preopen (preclosed) set of fbts Z. Hence gf is fuzzy pairwise strongly preopen (preclosed) mapping.∎

References

[1] K. K. Azad, *On fuzzy precontinuity, fuzzy almost continuity and fuzzy weakly continuity,* J.
Math. Anal. Appl.**82** (1981), 14-32.

[2] C. L. Chang, *Fuzzy topological spaces,* J.Math. Anal. Appl. **24** (1968), 182-190.

[3] A. Kandil, *Biproximaties and fuzzy bitopological spaces, Simon Steven* **63** (1989), 45-66.

[4] B. Krsteska, *Fuzzy strongly preopen sets and fuzzy strong precontinuity,* Mat. Vesnik, **50**. (1998), 111-123.

[5] S. Sampath Kumar, *Semiopen sets, semicontinuity and semiopen mappings in fuzzy bitopological spaces,* Fuzzy Sets and Systems, **64** (1994), 421-426.

[6] S. Sampath Kumar, *On fuzzy pairwise α - continuity and fuzzy pairwise precontinuity,* Fuzzy Sets and Systems, **62** (1994), 231-238.

[7] L. A. Zadeh, *Fuzzy sets,* Information and Control **8** (1965), 338-353.

Faculty of Mathematics and Natural Science
P.O. BOX 162
Skopje, Macedonia

e-mail address: biljanak@iunona.pmf.ukim.edu.mk

ФАЗИ ПОПАРНО ЈАКА НЕПРЕКИНАТОСТ

Билјана Крстеска

Апстракт. Во фази битополошки простори е воведен концептот на фази јако преотворени множества. Испитани се и проучени некои нивни основни својства. Во фази битополошки простори генерализарани се поимите фази јако прененепрекинати, фази јако преотворени и фази јако презатворени пресликувања. Испитани се и проучени некои нивни основни својстава и истите се доведени во врска со други послаби видови на фази попарно непрекинати пресликувања.