REDUCTION OF NONHOMOGENEOUS ORDINARY LINEAR DIFFERENTIAL EQUATION OF THIRD ORDER BASED ON A QUASIPERIODICITY OF THE SOLUTIONS

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Abstract. Using the condition (1) of quasi-periodicity of the functions we reduce the linear differential equation of third order to a linear differential equation of second order. Then, using the results at [1] and [2] we give some conditions of existence quasiperiodic solution with a constant quasiperiod for some linear differential equations of third order (4).
Key words: differential equation, quasiperiod, quasiperiodic solution.

1. INTRODUCTION

Definition 1.1. We say that $y=\varphi(x), x\in I\subseteq D_{\varphi}\subset R$ is a quasi-periodic function (QPF) if there are a function $\omega(x)$ and coefficient λ such that the relation

$$\varphi(x+\omega(x))=\lambda\varphi(x), \quad x, x+\omega(x)\in I . \tag{1}$$

is satisfied. The function $\omega(x)$ is called a quasi-period (QP) and λ is said to be a quasi-periodic coefficient (QPC) of the function $\varphi(x)$.

Remark 1.1. It holds generally $\lambda=\lambda(x,\omega(x))$ and in this case the existence of the relation (1) is very complicated problem. If $\omega(x)=\omega*=konst.$. and $\lambda=1$ for $x\in I$, then (1) is a definition of a periodic function in a classical sense. If $\omega=\omega(x)\neq konst.$ and $\lambda=1$ for $x,x+\omega\in I$, then (1) is a generalization of the definition to a periodic function and in this case $\omega=\omega(x)$ is a function of "repeating values" of $y=\varphi(x)$.

Suppose that the function $y(x)$ is given implicitly with the linear differential equation

$$F\left(x,y,y',\dots,y^{(n)},a(x),b(x),\dots,c(x)\right)=0 \tag{2}$$

where $a(x), b(x), \ldots, c(x) \in C_I^{(n-1)}$ at $I \subseteq D_a \cap D_b \cap \ldots \cap D_c \cap D_y$.

Let $y(x)$ be quasi-periodic solution (QPS) to (2), with a QP $\omega = \omega(x)$ and QPC λ, i.e.

$$y(x+\omega) = \lambda y(x)$$

where $\omega(x) \in C_I^n$, $\lambda > 0$, $\lambda \neq 1$ and x, $x+\omega(x) \in I$.

Using the following system (a reducible system)

$$\begin{cases} F(x, y, y', y'', \ldots, y^{(n)}, a(x), b(x), \ldots, c(x)) = 0 \\ F(x+\omega, y(x+\omega), y'(x+\omega), \ldots, y^{(n)}(x+\omega), a(x+\omega), \ldots, c(x+\omega)) = 0 \\ y(x+\omega(x)) = \lambda y(x) \\ \dfrac{d^m}{dx^m}(y(x+\omega(x))) = \lambda y^{(m)}(x), \quad m = 1, 2, \ldots, n \end{cases} \tag{3}$$

we reduce the equation (2) to the linear differential equation of (n-1) order of y. In the papers [1] and [2] , using the above procedure, we have given some conditions of existence QPS with a constant QP for the differential equations of first and second order and have found the form of the solutions. Here in a similar way we reduce a linear nonhomogeneous differential equation of third order to a differential equation of second order and give some conditions of existence QPS for this equation.

2. PRELIMINARY

Let

$$y''' + a(x)y'' + b(x)y' + c(x)y = d(x), \tag{4}$$

be a given differential equation, where $a(x)$, $b(x)$, $c(x)$, $d(x)$ are continuous and three times differentiable functions at $I \subseteq D_a \cap D_b \cap \ldots \cap D_c \cap D_y$ and $y = y(x)$ is QPS for (4), i.e.

$$y(x+\omega(x)) = \lambda y(x) \tag{5}$$

if $\omega(x) \in C_I^3$, $\lambda > 0$, $\lambda \neq 1$ and x, $x+\omega(x) \in I$.

In this case the reducible system (3) is:

$$\begin{cases} y'''(x) + a(x)y''(x) + b(x)y'(x) + c(x)y(x) = d(x) \\ y'''(x+\omega) + a(x+\omega)y''(x+\omega) + b(x+\omega)y'(x+\omega) + c(x+\omega)y(x+\omega) = d(x+\omega) \\ y(x+\omega) = \lambda y(x) \\ y'(x+\omega)(1+\omega') = \lambda y'(x) \\ y''(x+\omega)(1+\omega')^2 + y'(x+\omega)\omega'' = \lambda y''(x) \\ y'''(x+\omega)(1+\omega')^3 + 3y''(x+\omega)(1+\omega')\omega'' + y'(x+\omega)\omega''' = \lambda y'''(x) \end{cases} \tag{6}$$

If $1+\omega'\neq 0$, i.e. $\omega\neq -x+k, k\in R$, we can eliminate $y(x+\omega)$ and its derivatives $y'''(x), y'(x+\omega), y''(x+\omega), y'''(x+\omega)$ and so we reduce the equation (4) to the equation

$$\begin{cases}\lambda y''\left[a(x+\omega)(1+\omega')^2-a(x)(1+\omega')-3\omega''\right](1+\omega')+\\+\lambda y'\left[b(x+\omega)(1+\omega')^4-b(x)(1+\omega')^2-a(x+\omega)(1+\omega')^2\omega''-3\omega''^2-(1+\omega')\omega'''\right]+\\+\lambda y\left[c(x+\omega)(1+\omega')^3-c(x)\right](1+\omega')^2-\left[d(x+\omega)(1+\omega')^3-\lambda d(x)\right](1+\omega')^2=0\end{cases} \quad (7)$$

After the above arguments we have a proof of the following theorem.

Theorem 2.1. Let the nonhomogeneous linear differential equation of third order (4) have QPS defined by the relation (5). Using the system (6), the equation (4) is reduced to a nonhomogeneous linear differential equation of second order given by (7).

3. EXISTENCE OF QPS WITH A CONSTANT QP TO THE EQUATION (4).

Lemma 3.1. Let $y(x)$ be QPS to (4) with a constant QP $\omega(x)=const.=\varpi$ and a QPC λ ($\lambda>0$, $\lambda\neq 1$). Then it holds

$$\begin{aligned}&\lambda y''[a(x+\varpi)-a(x)]+\lambda y'\cdot[b(x+\varpi)-b(x)]+\lambda y[c(x+\varpi)-c(x)]-\\&-[d(x+\varpi)-\lambda d(x)]=0\end{aligned} \quad (8)$$

Proof. Substituting $\omega=\varpi, \varpi'=\varpi''=\varpi'''=0$ in (7) we obtain (8). ▪

Theorem 3.1. Let the coefficients $a(x)$, $b(x)$, $c(x)$, $d(x)$ be QPF with a constant QP ϖ and QPC μ,ν,η,ς respectively, such that $\mu\neq\nu$, $\nu\neq\eta,\mu\neq\eta,\mu\neq\varsigma,\nu\neq\varsigma$, μ, $\nu,\eta,\varsigma\neq\lambda$. The equation (4) has QPS $y(x)$ with a QP ϖ and a QPC λ ($\lambda>0,\lambda\neq 1$), if the relations

3.1.1. $\left(\frac{d}{c}\right)''' + \frac{\mu-\nu}{\mu-1}\cdot\left(\frac{d}{c}\right)'\cdot b=0$, and

3.1.2. $\left(\frac{d}{c}\right)''' - \frac{\mu-\nu}{\nu-1}\cdot\left(\frac{d}{c}\right)''\cdot a=0$

are satisfied. Then the QPS of (4) has a form $y=\dfrac{d(x)}{c(x)}$.

Proof. Using the conditions

$$\begin{aligned}&a(x+\varpi)=\mu a(x),\ b(x+\varpi)=\nu b(x),\\&c(x+\varpi)=\eta c(x),\ d(x+\varpi)=\varsigma d(x),\\&\mu\neq\nu\ \ \nu\neq\eta,\mu\neq\eta,\mu\neq\varsigma,\nu\neq\varsigma\ \mu,\ \nu,\eta,\varsigma\neq\lambda\end{aligned} \quad (9)$$

and Lemma 3.1. we reduce the equation (4) to the equation

$$\lambda(\mu-1)a(x)y''+\lambda(\nu-1)b(x)y'+\lambda(\eta-1)c(x)y-(\varsigma-\lambda)d(x)=0 \qquad (10)$$

Depending on μ, ν,η,ς the following cases are possible ([1],[2]):

a) If $\mu=\nu=1,\eta\neq 1$, i.e. $a(x)$ and $b(x)$ are periodic functions with a period ϖ, then (10) is equivalent with the equation

$$\lambda(\eta-1)c(x)y-(\varsigma-\lambda)d(x)=0,$$

whose solution is

$$y=\frac{\varsigma-\lambda}{\lambda(\eta-1)}\cdot\frac{d(x)}{c(x)},\ c(x)\neq 0 \qquad (11)$$

The solution (11) is a QPF with a QP ϖ and a QPC $\lambda=\frac{\varsigma}{\eta}$ and

$$\frac{\varsigma-\lambda}{\lambda(\eta-1)}=\frac{\varsigma-\frac{\varsigma}{\eta}}{\frac{\varsigma}{\eta}(\eta-1)}=1$$

b) If $\mu-1\neq 0$, then we can write the equation (10) in the form

$$y''+f(x)y'+g(x)y-h(x)=0 \qquad (12)$$

where $f(x)=\frac{\nu-1}{\mu-1}\cdot\frac{b(x)}{a(x)}$, $g(x)=\frac{\eta-1}{\mu-1}\cdot\frac{c(x)}{a(x)}$, $h(x)=\frac{\varsigma-\lambda}{\lambda(\mu-1)}\cdot\frac{d(x)}{a(x)}$, According to the conditions of the theorem, all of the functions $f(x)$, $g(x)$, $h(x)$ are QPF with a QP ϖ a and QPC $\nu_1=\frac{\nu}{\mu},\eta_1=\frac{\eta}{\mu}$, $\varsigma_1=\frac{\varsigma}{\mu}$, such that

$\nu_1\neq\eta_1,\eta_1\neq\varsigma_1,\nu_1\neq\varsigma_1,\varsigma_1\neq\lambda$ and $\nu_1\neq 1$. In this case (12) also has a QPS

$$y=\frac{\varsigma_1-\lambda}{\lambda(\eta_1-1)}\cdot\frac{h(x)}{g(x)}=\frac{d(x)}{c(x)} \qquad (11')$$

with a QP ϖ and a QPC $\lambda=\frac{\varsigma_1}{\eta_1}=\frac{\varsigma}{\eta}$ and the relation

$$\left(\frac{d}{c}\right)''+\frac{\nu-1}{\mu-1}\cdot\left(\frac{d}{c}\right)'\frac{b}{a}=0 \quad \left(a=a(x),\ b=b(x),\ c=c(x),\ d=d(x)\right) \qquad (13)$$

is satisfied .

Since the solution y, determined by (11), i.e. (11′), is also a solution of (4), the relation

$$\left(\frac{d}{c}\right)'''+a\cdot\left(\frac{d}{c}\right)''+b\cdot\left(\frac{d}{c}\right)'=0 \qquad (14)$$

has to be satisfied. From (13) and (14), after short transformation we obtain the relations 3.1.1. and 3.1.2.

Remark 3.1. It should be noted that, under the conditions of the theorem, the quasi-periodicity of the solution (11) does not depend on the coefficients $a(x)$ and $b(x)$ for (4). Namely, the solution (11) for (4) is a QPF in the both cases, if $a(x)$ and $b(x)$ are periodic or QPF . ▪

Example 3.1. The equation

$$y''' + a(x)y'' + b(x)y' + e^{-x}y = e^{x}\sin x$$

where $a(x)$ and $b(x)$ are

$$a(x) = \frac{2\sin x + 11\cos x}{3\sin x + 4\cos x}, \qquad b(x) = -\frac{2(2\sin x + 11\cos x)}{2\sin x + \cos x},$$

satisfies the conditions of the Theorem 3.1. and has a QPS

$$y = e^{2x}\sin x$$

with a QP $\varpi = 2\pi$ and QPC $\lambda = e^{4\pi}$.

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РЕДУЦИБИЛНОСТ НА НЕХОМОГЕНА ЛИНЕАРНА ДИФЕРЕНЦИЈАЛНА РАВЕНКА ОД III РЕД НА БАЗА НА КВАЗИПЕРИОДИЧНОСТ НА РЕШЕНИЈАТА

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Апстракт. Диференцијалната равенка од III ред (4), што има квазипериодични решенија, со помош на редукциониот систем (6) ја редуцираме во равенка од II ред која, во случај кога коефициентите во (4) се квазипериодични функции со константен квазипериод, се сведува на равенка од вид (10). Главниот резултат во трудот е даден во Теорема 3.1.