VOLUME OF SOME CLOSED HYPERSOLIDS

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Abstract. In this paper it is found general formula for volume of one class of closed solids bounded by hypersurfaces in n-dimensional Euclidean space.

1. Main result

First we will find the volume of the solid bounded by the hypersurface

(1) $$\sum_{i=1}^{n} \left| \frac{x_i}{a_i} \right|^{\alpha_i} = 1, \qquad (\alpha_i > 0, a_i \neq 0; 1 \leq i \leq n).$$

Using the changes of the variables $x_1, \cdots, x_n$ by

$$x_1 = a_1 \cdot (\sin t_1)^{2/\alpha_1},$$

$$x_2 = a_2 \cdot (\cos t_1)^{2/\alpha_2} \cdot (\sin t_2)^{2/\alpha_2},$$

(2) $$x_3 = a_3 \cdot (\cos t_1)^{2/\alpha_3} \cdot (\cos t_2)^{2/\alpha_3} \cdot (\sin t_3)^{2/\alpha_3},$$

$$\vdots$$

$$x_{n-1} = a_{n-1} \cdot (\cos t_1)^{2/\alpha_{n-1}} \cdots (\cos t_{n-2})^{2/\alpha_{n-1}} \cdot (\sin t_{n-1})^{2/\alpha_{n-1}},$$

$$x_n = a_n \cdot (\cos t_1)^{2/\alpha_n} \cdots (\cos t_{n-1})^{2/\alpha_n},$$

then the Jacobian is given by

(3) $$J = \frac{\partial(x_2, \cdots, x_n)}{\partial(t_1, \cdots, t_{n-1})} =$$

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$$=\begin{vmatrix} \frac{\partial x_2}{\partial t_1} & \frac{\partial x_2}{\partial t_2} & 0 & 0 & \cdots & 0 & 0 \\ \frac{\partial x_3}{\partial t_1} & \frac{\partial x_3}{\partial t_2} & \frac{\partial x_3}{\partial t_3} & 0 & \cdots & 0 & 0 \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \frac{\partial x_{n-2}}{\partial t_1} & \frac{\partial x_{n-2}}{\partial t_2} & \frac{\partial x_{n-2}}{\partial t_3} & \frac{\partial x_{n-2}}{\partial t_4} & \cdots & \frac{\partial x_{n-2}}{\partial t_{n-2}} & 0 \\ \frac{\partial x_{n-1}}{\partial t_1} & \frac{\partial x_{n-1}}{\partial t_2} & \frac{\partial x_{n-1}}{\partial t_3} & \frac{\partial x_{n-1}}{\partial t_4} & \cdots & \frac{\partial x_{n-1}}{\partial t_{n-2}} & \frac{\partial x_{n-1}}{\partial t_{n-1}} \\ \frac{\partial x_n}{\partial t_1} & \frac{\partial x_n}{\partial t_2} & \frac{\partial x_n}{\partial t_3} & \frac{\partial x_n}{\partial t_4} & \cdots & \frac{\partial x_n}{\partial t_{n-2}} & \frac{\partial x_n}{\partial t_{n-1}} \end{vmatrix} =$$

$$= 2^{n-1}\Big(\prod_{i=2}^{n}\frac{a_i}{\alpha_i}\Big)\sin t_1\cdot(\cos t_1)^{2(\alpha_2^{-1}+\cdots+\alpha_n^{-1})-1}\cdot(\sin t_2)^{2\alpha_2^{-1}-1}\times$$

$$\times(\cos t_2)^{2(\alpha_3^{-1}+\cdots+\alpha_n^{-1})-1}\cdots(\sin t_{n-1})^{2\alpha_{n-1}^{-1}-1}\cdot(\cos t_{n-1})^{2\alpha_n^{-1}-1}\cdot detA,$$

where A is orthogonal matrix, such that $\mid detA \mid = 1$.

Since the solid is symmetric with respect to each hyperplane $x_i = 0$ $(1 \le i \le n)$, it is is sufficient to find the volume in the first quadrant and then to multiply by 2^n. Note that under the above coordinate transformation, the set $R = \{(x_2, x_3, ..., x_n) \mid \left|\frac{x_2}{a_2}\right|^{\alpha_2} + \left|\frac{x_3}{a_3}\right|^{\alpha_3} + ... + \left|\frac{x_n}{a_n}\right|^{\alpha_n} \le 1\}$ maps into the set $R' = \{(t_1, t_2, ..., t_{n-1}) \mid 0 \le t_i \le \pi/2\} = [0, \pi/2]^{n-1}$. Namely,

$$\left|\frac{x_2}{a_2}\right|^{\alpha_2} + \left|\frac{x_3}{a_3}\right|^{\alpha_3} + ... + \left|\frac{x_n}{a_n}\right|^{\alpha_n} = \cos^2 t_1 \sin^2 t_2 + \cos^2 t_1 \cos^2 t_2 \sin^2 t_3 + \ldots$$

$\ldots + \cos^2 t_1 \cos^2 t_2 \cdots \cos^2 t_{n-2} \sin^2 t_{n-1}$ and

$$\cos^2 t_1 \sin^2 t_2 + \cos^2 t_1 \cos^2 t_2 \sin^2 t_3 + \ldots + \cos^2 t_1 \cos^2 t_2 \cdots \cos^t_{n-2} \cos^2 t_{n-1} \le$$

$$\le \cos^2 t_1 \sin^2 t_2 + \cos^2 t_1 \cos^2 t_2 \sin^2 t_3 + \ldots + \cos^2 t_1 \cos^2 t_2 \cdots \cos^2 t_{n-2} =$$

$= \cos^2 t_1 \le 1$ for each $t_1, t_2, \cdots, t_n \in [0, \pi/2]$. Moreover, for each $(x_2, x_3, ..., x_n) \in R$, there is an unique $(t_1, t_2, \cdots, t_{n-1}) \in [0, \pi/2]^{n-1}$ such that the last $n-1$ equations of (2) are satisfied. Namely, if we choose $x_1 \ge 0$ such that (1) is satisfied, then x_1 must be of the form $x_1 = a_1 \cdot (\sin t_1)^{2/\alpha_1}$, and hence t_1 is uniquely determined. Further, t_2 is uniquely determined, t_3 is uniquely determined and so on. Also, is it is obvious that each $(t_1, t_2, \cdots, t_{n-1}) \in [0, \pi/2]^{n-1}$ determines an (n-1)-tuple $(x_2, x_3, \ldots, x_n)$.

Hence the volume of the hypersolid is

$$V_n = \left|\int_R \cdots \int x_1 dx_2 \dots dx_n\right| = \tag{4}$$

$$= 2^n \left|\int_{R'} \cdots \int a_1 (\sin t_1)^{2/\alpha_1} \mid J \mid dt_1 \dots dt_{n-1}\right| =$$

$$= 2^n \cdot 2^{n-1} \Big(\prod_{i=1}^n a_i\Big)\Big(\prod_{i=2}^n \frac{1}{\alpha_i}\Big)\left|\int_0^{\pi/2} (\sin t_1)^{2\alpha_1^{-1}+1} \cdot (\cos t_1)^{2(\alpha_2^{-1}+\cdots+\alpha_n^{-1})-1} dt_1 \times\right.$$

$$\times \int_0^{\pi/2} (\sin t_2)^{2\alpha_2^{-1}-1} \cdot (\cos t_2)^{2(\alpha_3^{-1}+\cdots+\alpha_n^{-1})-1} dt_2 \times \cdots \times$$

$$\left.\times \int_0^{\pi/2} (\sin t_{n-1})^{2\alpha_{n-1}^{-1}-1} \cdot (\cos t_{n-1})^{2\alpha_n^{-1}-1} dt_{n-1}\right|.$$

Using the gamma function [1, p. 1-5], we obtain

$$V_n = 2^n \cdot 2^{n-1} \Big(\prod_{i=1}^n a_i\Big)\Big(\prod_{i=2}^n \frac{1}{\alpha_i}\Big) \cdot \frac{1}{2} \cdot \frac{\Gamma(\frac{1}{\alpha_1}+1)\Gamma(\frac{1}{\alpha_2}+\cdots+\frac{1}{\alpha_n})}{\Gamma(\frac{1}{\alpha_1}+\cdots+\frac{1}{\alpha_n})} \times$$

$$\times \frac{1}{2} \cdot \frac{\Gamma(\frac{1}{\alpha_2})\Gamma(\frac{1}{\alpha_3}+\cdots+\frac{1}{\alpha_n})}{\Gamma(\frac{1}{\alpha_2}+\cdots+\frac{1}{\alpha_n})} \cdots \frac{1}{2} \cdot \frac{\Gamma(\frac{1}{\alpha_{n-1}})\Gamma(\frac{1}{\alpha_n})}{\Gamma(\frac{1}{\alpha_{n-1}}+\frac{1}{\alpha_n})},$$

i.e.

$$V_n = \frac{2^n \cdot \prod_{i=1}^n a_i}{\Big(\prod_{i=1}^n \alpha_i\Big)\Big(\sum_{i=1}^n \frac{1}{\alpha_i}\Big)} \cdot \frac{\prod_{i=1}^n \Gamma\Big(\frac{1}{\alpha_i}\Big)}{\Gamma\Big(\sum_{i=1}^n \frac{1}{\alpha_i}\Big)}, \tag{5}$$

where n is a positive integer representing the dimension of the space.

Remark 1. As special cases of (5) we obtain:

1^o. If $\alpha_i = 1, (1 \le i \le n)$, then the volume of the polyhedron [2, exe. 42, p. 130] is

$$V_n = \frac{2^n}{n!}\Big(\prod_{i=1}^n a_i\Big).$$

2^o. If $\alpha_i = 2, (1 \leq i \leq n)$, then the volume of the hyperellipsoid [3] is

$$V_n = \frac{(\pi)^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}\left(\prod_{i=1}^{n} a_i\right).$$

3^o. If $\alpha_i = 2, a_i = r; (1 \leq i \leq n)$, then the volume of the hypersphere [4, p. 136] is

$$V_n = \frac{(\pi)^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} r^n.$$

4^o. If $\alpha_i = 2/3, a_i = r; (1 \leq i \leq n)$, then the volume of the hyperastroid is

$$V_n = \frac{3^n(\pi)^{n/2}}{2^n\Gamma\left(\frac{3n}{2}+1\right)} r^n.$$

Remark 2. In [5] are obtained the following particular cases of (5):

1^o. $\alpha_1 = \alpha_2 = \alpha_3 = n$, exe. 97, p. 526,

2^o. $\alpha_1 = \alpha_2 = 2, \alpha_3 = 4$, exe. 163, p. 574,

3^o. $\alpha_1 = \alpha_2 = \alpha_3 = 1/2$, exe. 169, p. 577,

4^o. $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$, exe. 170, p. 578,

5^o. $\alpha_1 = \alpha_2 = \alpha_3 = 2/3$, exe. 171, p. 578,

6^o. $\alpha_1 = m, \alpha_2 = n, \alpha_3 = p$, exe. 183, p. 585.

2. Application of the main result

Now we can find the area $A(\mathbf{p})$ of the orthogonal projection of the hypersolid

$$(6) \qquad \sum_{i=1}^{n} \left| \frac{x_i}{a_i} \right|^{\alpha} = 1, \qquad (\alpha > 0, a_i \neq 0; 1 \leq i \leq n)$$

over the hyperplane orthogonal to the unit vector $\mathbf{p} = (p_1, p_2, \cdots, p_n)$.

Let V be the volume of the cylinder C having the projection $A(\mathbf{p})$ as a basis and the vector $\mathbf{p}$ as a generatrise. Then it holds $A(\mathbf{p}) = V$. Let us consider the following linear transformation $\mathbf{x} \to \mathbf{x}' = T\mathbf{x}$ in R^n, given by the matrix $T = diag(a_1^{-1}, a_2^{-1}, \cdots, a_n^{-1})$. The hypersolid (6) maps into $|\, \mathbf{x}' \,|_{\alpha} = 1$, the hypersolid C maps into C' with volume V' and the vector $\mathbf{p}$ maps into $\mathbf{p}' = T\mathbf{p}$.

Thus, it holds

$$(7) \qquad A(\mathbf{p}) = V = V' \cdot a_k \cdot \mid \mathbf{p}' \mid_\alpha, \qquad (1 \leq k \leq n)$$

where V' is the $(n-1)$-dimensional volume of the projection of the hypersolid (6) projected over each hyperplane $x_k = 0$, $(1 \leq k \leq n)$ given by

$$(8) \qquad V' = \frac{2^{n-1}\left[\Gamma\left(\frac{1}{\alpha}\right)\right]^{n-1}}{\alpha^{n-1}\Gamma\left(\frac{n-1}{\alpha}+1\right)} \cdot \frac{1}{a_k}\prod_{i=1}^{n} a_i, \qquad (1 \leq k \leq n)$$

in according with (5).

The required result follows just from (7) and (8)

$$(9) \quad A(\mathbf{p}) = \frac{2^{n-1}\left[\Gamma\left(\frac{1}{\alpha}\right)\right]^{n-1}\left(\prod_{i=1}^{n} a_i\right)}{\alpha^{n-1}\Gamma\left(\frac{n-1}{\alpha}+1\right)} \cdot \left[\sum_{i=1}^{n}\left(\frac{p_i}{a_i}\right)^{\alpha}\right]^{1/\alpha}, \qquad (n \in \mathbf{N}).$$

The above equality is generalization of Helmbond's problem [6].

3. An expanded result

It is of interest to find the volume of the hypersolid bounded by the hypersurface

$$(10) \qquad \sum_{i=1}^{n}\left|\frac{1}{A_i}\sum_{j=1}^{n} a_{ij}x_j\right|^{\alpha_i} = 1, \qquad (\alpha_i > 0, A_i \neq 0; 1 \leq i \leq n).$$

if

$$(11) \qquad \Delta_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0.$$

Introducing the changes

$$(12) \qquad \sum_{j=1}^{n} a_{ij}x_j = t_i, \quad (1 \leq i \leq n)$$

the equality (10) becomes

$$\sum_{i=1}^{n} \left| \frac{t_i}{A_i} \right|^{\alpha_i} = 1, \qquad (\alpha_i > 0, A_i \neq 0; 1 \leq i \leq n) \tag{13}$$

and the Jacobian of the transformation (10) becomes

$$J = \frac{\partial(x_1, \cdots, x_n)}{\partial(u_1, \cdots, u_n)} = \left[\frac{\partial(u_1, \cdots, u_n)}{\partial(x_1, \cdots, x_n)}\right]^{-1} = \frac{1}{\Delta_n}. \tag{14}$$

Analogously to the first case, the volume of the hypersolid bounded by the hypersurface (10), is given by

$$V_n = \left| \int_R \cdots \int dx_1 \cdots dx_n \right| = 2^n \left| \int_{R'} \cdots \int \mid J \mid dt_1 \cdots dt_n \right| = \tag{15}$$

$$= \frac{2^n}{\Delta_n} \left| \int_{R'} \cdots \int dt_1 \cdots dt_n \right|.$$

This multiple integral is previously solved, and thus using (5), we obtain the requested volume to be

$$V_n = \frac{2^n \cdot \prod_{i=1}^{n} A_i}{\Delta_n \left(\prod_{i=1}^{n} \alpha_i\right) \left(\sum_{i=1}^{n} \frac{1}{\alpha_i}\right)} \cdot \frac{\prod_{i=1}^{n} \Gamma\left(\frac{1}{\alpha_i}\right)}{\Gamma\left(\sum_{i=1}^{n} \frac{1}{\alpha_i}\right)}. \tag{16}$$

Remark 3. We obtain from (16) the following particular case in [5]:
1^o. $\alpha_i = 2$, $A_i = a$; $(i = 1, 2, 3)$ exe. 181, p. 584.

4. Conclusion

Here presented results complete the well-known books of Fihtengol'c [7, p. 391-413], Shilov [8, p. 243-250] and Mitrinović [9, p. 118-138].

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ВОЛУМЕН НА НЕКОИ ТЕЛА ОГРАНИЧЕНИ СО ХИПЕРПОВРШИНИ

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Апстракт: На почетокот од трудот е докажано дека волуменот на телото ограничено со хиперповршината (1) е зададена со формулата (5). Добиената општа формула содржи како специјални случаи повеќе формули познати во литературата. Потоа како примена на тој резултат дадена е генерализација на Хелмбондовиот проблем. На крај, за волуменот на телото ограничено со хиперповршината (10) добиена е формулата (16).

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