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$L^1(\mu)$ as a *n*-Normed Space

Abstract. *The concept of a* n-*norm on the vector space of dimension greater or equal to* n, n>*1, introduced by A.Misiak ([4]), is a multi-dimensional anaoque of the concept of the norm. In [1], [2], [3] and [4] several properties of the* n-*normed spaces are proved. In this work we will proved that if* μ *is a positive measure on an arbitrary measurable space* X *and* $L^1(\mu)$ *is the space of real measurable functions* f *on* X *such that* $\int_X |f| d\mu < \infty$, *then in* $L^1(\mu)$ *can be introduced a* n-*norm.*

Let L be a real vector space with dimension greater of n, $n \geq 1$ and $\|\bullet,\ldots,\bullet\|$ be a real function on L^n with the following properties:

i) $\|x_1,\ldots,x_n\| \geq 0$, for every $x_1,\ldots,x_n \in L$ and $\|x_1,\ldots,x_n\| = 0$ if and only if the set $\{x_1,\ldots,x_n\}$ is linearly dependent.

ii) $\|x_1,\ldots,x_n\| = \|\pi(x_1),\ldots,\pi(x_n)\|$, for every $x_1,\ldots,x_n \in L$ and every bejection $\pi:\{x_1,\ldots,x_n\} \to \{x_1,\ldots,x_n\}$

iii) $\|\alpha x_1,\ldots,x_n\| = |\alpha| \cdot \|x_1,\ldots,x_n\|$, for every $x_1,\ldots,x_n \in L$ and every $\alpha \in R$

iv) $\left\|x_1 + x_1',\ldots,x_n\right\| \leq \|x_1,\ldots,x_n\| + \left\|x_1',\ldots,x_n\right\|$, for every $x_1,\ldots,x_n,x_1' \in L$.

The function $\|\bullet,\ldots,\bullet\|$ is called a n-norm on L, and $(L^n,\|\bullet,\ldots,\bullet\|)$ is called n-normed space.

Let X be an arbitrary measurable space and μ be a positive measure on X. We denote $L^1(\mu)$ the set of all real μ measurable functions f on X, such that $\int_X |f| d\mu < \infty$. The set $L^1(\mu)$ with usual opperations adding of functions and product of a function with a real number is a real vector space (Theorem 1.32, [6]).

Lemma 1. Let $n \geq 2$. The vectors $f_1,...,f_n \in L^1(\mu)$ are linearly dependent if and only if for every measurable sets $E_1,...,E_n$ we have:

$$\begin{vmatrix} \int_{E_1} f_1 d\mu & \int_{E_2} f_1 d\mu & \dots & \int_{E_{n-1}} f_1 d\mu & \int_{E_n} f_1 d\mu \\ \int_{E_1} f_2 d\mu & \int_{E_2} f_2 d\mu & \dots & \int_{E_{n-1}} f_2 d\mu & \int_{E_n} f_2 d\mu \\ \dots & \dots & \dots & \dots & \dots \\ \int_{E_1} f_n d\mu & \int_{E_2} f_n d\mu & \dots & \int_{E_{n-1}} f_n d\mu & \int_{E_n} f_n d\mu \end{vmatrix} = 0. \qquad (1)$$

Proof. It is clear that if the vectors $f_1,...,f_n \in L^1(\mu)$ are lineary dependent, then for every measurable sets $E_1,...,E_n$ (1) is true.

We will prove the converse by induction in n.

Let $n = 2$. Let for $f_1, f_2 \in L^1(\mu)$ and for every measurable sets E_1, E_2 it is true

$$\begin{vmatrix} \int_{E_1} f_1 d\mu & \int_{E_2} f_1 d\mu \\ \int_{E_1} f_2 d\mu & \int_{E_2} f_2 d\mu \end{vmatrix} = 0,$$

which means

$$\int_{E_1} f_1 d\mu \int_{E_2} f_2 d\mu - \int_{E_2} f_1 d\mu \int_{E_1} f_2 d\mu = 0. \qquad (2)$$

We will consider two cases:

i) For every measurable set E, $\int_E f_1 d\mu = 0$. Then $f_1 = 0$, a.e. on X (Theorem 1.39 b), [6]),and so $f_1 = 0 \cdot f_2$. This means that f_1 and f_2 are lineary dependent.

ii) There exist measurable set E such that $\int_E f_1 d\mu \neq 0$. Let

$$\alpha = \frac{\int_E f_2 d\mu}{\int_E f_1 d\mu}.$$

From (2) follows that for every measurable set $E^{'}$ it is true

$$\int_{E^{'}} (f_2 - \alpha f_1) d\mu = 0,$$

which means that $f_1 - \alpha \cdot f_2 = 0$, a.e. on X and so f_1 and f_2 are lineary dependent

Suppose that the statement is true for some $n \geq 2$ and that the vectors $f_1, \ldots, f_n, f_{n+1} \in L^1(\mu)$ and that for every measurable sets $E_1, \ldots, E_n, E_{n+1}$ it is true

$$\begin{vmatrix} \int_{E_1} f_1 d\mu & \int_{E_2} f_1 d\mu & \ldots & \int_{E_n} f_1 d\mu & \int_{E_{n+1}} f_1 d\mu \\ \int_{E_1} f_2 d\mu & \int_{E_2} f_2 d\mu & \ldots & \int_{E_n} f_2 d\mu & \int_{E_{n+1}} f_2 d\mu \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \int_{E_1} f_{n+1} d\mu & \int_{E_2} f_{n+1} d\mu & \ldots & \int_{E_n} f_{n+1} d\mu & \int_{E_{n+1}} f_{n+1} d\mu \end{vmatrix} = 0.$$

which means

$$\sum_{i=1}^{n+1} (-1)^{n+1+i} \int_{E_{n+1}} f_i d\mu \begin{vmatrix} \int_{E_1} f_1 d\mu & \int_{E_2} f_1 d\mu & \ldots & \int_{E_n} f_1 d\mu \\ \ldots & \ldots & \ldots & \ldots \\ \int_{E_1} f_{i-1} d\mu & \int_{E_2} f_{i-1} d\mu & \ldots & \int_{E_n} f_{i-1} d\mu \\ \int_{E_1} f_{i+1} d\mu & \int_{E_2} f_{i+1} d\mu & \ldots & \int_{E_n} f_{i+1} d\mu \\ \ldots & \ldots & \ldots & \ldots \\ \int_{E_1} f_{n+1} d\mu & \int_{E_2} f_{n+1} d\mu & \ldots & \int_{E_n} f_{n+1} d\mu \end{vmatrix} = 0. \quad (3)$$

We will consider two cases:

i) For every measurable sets $E_1, \ldots, E_n, E_{n+1}$ all determinants in (3) are equal to zero. Then for every subset of $\{f_1, \ldots, f_n, f_{n+1}\}$ with n-elements the condition (1) is satisfied. By the inductive assumptation this subsets are lineary dependent and hence the set $\{f_1,...,f_n,f_{n+1}\}$ is lineary dependent.

ii) There exist measurable sets $E_1,...,E_n,E_{n+1}$ such that one of the determinants in (3) is different of zero. Without loosing the generality we may assume that

$$\begin{vmatrix} \int\limits_{E_1} f_1 d\mu & \int\limits_{E_2} f_1 d\mu & \dots & \int\limits_{E_{n-1}} f_1 d\mu & \int\limits_{E_n} f_1 d\mu \\ \int\limits_{E_1} f_2 d\mu & \int\limits_{E_2} f_2 d\mu & \dots & \int\limits_{E_{n-1}} f_2 d\mu & \int\limits_{E_n} f_2 d\mu \\ \dots & \dots & \dots & \dots & \dots \\ \int\limits_{E_1} f_n d\mu & \int\limits_{E_2} f_n d\mu & \dots & \int\limits_{E_{n-1}} f_n d\mu & \int\limits_{E_n} f_n d\mu \end{vmatrix} \neq 0.$$

Put for $i = 1,...,n$

$$\alpha_i = (-1)^{n+i} \frac{\begin{vmatrix} \int\limits_{E_1} f_1 d\mu & \int\limits_{E_2} f_1 d\mu & \dots & \int\limits_{E_n} f_1 d\mu \\ \dots & \dots & \dots & \dots \\ \int\limits_{E_1} f_{i-1} d\mu & \int\limits_{E_2} f_{i-1} d\mu & \dots & \int\limits_{E_n} f_{i-1} d\mu \\ \int\limits_{E_1} f_{i+1} d\mu & \int\limits_{E_2} f_{i+1} d\mu & \dots & \int\limits_{E_n} f_{i+1} d\mu \\ \dots & \dots & \dots & \dots \\ \int\limits_{E_1} f_{n+1} d\mu & \int\limits_{E_2} f_{n+1} d\mu & \dots & \int\limits_{E_n} f_{n+1} d\mu \end{vmatrix}}{\begin{vmatrix} \int\limits_{E_1} f_1 d\mu & \int\limits_{E_2} f_1 d\mu & \dots & \int\limits_{E_n} f_1 d\mu \\ \int\limits_{E_1} f_2 d\mu & \int\limits_{E_2} f_2 d\mu & \dots & \int\limits_{E_n} f_2 d\mu \\ \dots & \dots & \dots & \dots \\ \int\limits_{E_1} f_2 d\mu & \int\limits_{E_2} f_n d\mu & \dots & \int\limits_{E_n} f_n d\mu \end{vmatrix}}, \tag{4}$$

Fix the sets $E_1,...,E_n$. Then for every measurable set E it is true:

$$\begin{vmatrix} \int\limits_{E_1} f_1 d\mu & \int\limits_{E_2} f_1 d\mu & \dots & \int\limits_{E_n} f_1 d\mu & \int\limits_{E} f_1 d\mu \\ \int\limits_{E_1} f_2 d\mu & \int\limits_{E_2} f_2 d\mu & \dots & \int\limits_{E_n} f_2 d\mu & \int\limits_{E} f_2 d\mu \\ \dots & \dots & \dots & \dots & \dots \\ \int\limits_{E_1} f_{n+1} d\mu & \int\limits_{E_2} f_{n+1} d\mu & \dots & \int\limits_{E_n} f_{n+1} d\mu & \int\limits_{E} f_{n+1} d\mu \end{vmatrix} = 0,$$

which implies

$$\int_E (f_{n+1} - \sum_{i=1}^{n} \alpha_i f_i) d\mu = 0,$$

where $\alpha_i \;\; i = 1,\dots,n$ are defined as in (4).Hence, $f_{n+1} = \sum_{i=1}^{n} \alpha_i f_i$, a.e. on X.

Define a function $\|\bullet,\dots,\bullet\| : L^1(\mu)$ h ... h $L^1(\mu) \to R$ by

$$\|f_1,\dots,f_n\| = \sup_{E_1,\dots E_n} \left| \begin{vmatrix} \int_{E_1} f_1 d\mu & \int_{E_2} f_1 d\mu & \dots & \int_{E_{n-1}} f_1 d\mu & \int_{E_n} f_1 d\mu \\ \int_{E_1} f_2 d\mu & \int_{E_2} f_2 d\mu & \dots & \int_{E_{n-1}} f_2 d\mu & \int_{E_n} f_2 d\mu \\ \dots & \dots & \dots & \dots & \dots \\ \int_{E_1} f_n d\mu & \int_{E_2} f_n d\mu & \dots & \int_{E_{n-1}} f_n d\mu & \int_{E_n} f_n d\mu \end{vmatrix} \right|. \quad (5)$$

By Theorem 1.33, [6], for every measurable set E it is true

$$|\int_E f_i d\mu| \le \int_E |f_i| d\mu \le \int_X |f_i| d\mu = M_i < \infty, \text{ za } i = 1,\dots,n,$$

which implies $\|f_1,\dots,f_n\| \le n! \prod_{i=1}^{n} M_i$. Hence, the function $\|\bullet,\dots,\bullet\|$ is good defined.

Lemma 2. $(L^1(\mu), \|\bullet,\dots,\bullet\|)$ is a real n-normed space.

Proof. Since the function $\|\bullet,\dots,\bullet\|$ is good defined, it is enough to prove that it is satisfy the axioms of the n-norm. We have:

i) By the definition of $\|\bullet,\dots,\bullet\| : L^1(\mu)$ h ... h $L^1(\mu) \to R$ follows that $\|f_1,\dots,f_n\| \ge 0$. It is clear that $\|f_1,\dots,f_n\| = 0$ if and only if

$$\begin{vmatrix} \int_{E_1} f_1 d\mu & \int_{E_2} f_1 d\mu & \dots & \int_{E_{n-1}} f_1 d\mu & \int_{E_n} f_1 d\mu \\ \int_{E_1} f_2 d\mu & \int_{E_2} f_2 d\mu & \dots & \int_{E_{n-1}} f_2 d\mu & \int_{E_n} f_2 d\mu \\ \dots & \dots & \dots & \dots & \dots \\ \int_{E_1} f_n d\mu & \int_{E_2} f_n d\mu & \dots & \int_{E_{n-1}} f_n d\mu & \int_{E_n} f_n d\mu \end{vmatrix} = 0$$

for every measurable sets $E_1,...,E_n$. By Lemma 1 $\|f_1,...,f_n\|=0$ if and only if the vectors $f_1,...,f_n$ are lineary dependent.

The validity of the other axioms of the n-norm follows from the properties of the determinant and the supremum.

In the end of this work we will give several notes for the space $L^1(\mu)$.

1. In [1] is proved that in every n-normed space can by define a topology τ which make the space into a local convex space and in this topology the n-norm is continuous in each variable. We have the question: if in $L^1(\mu)$ we introduce a topology τ as above described way, what properties has the space $(L^1(\mu),\tau)$.

2. Let L be a real vector space and $x_1,x_2,...,x_{n-1} \in L$. Denote by $P(x_1,x_2,...,x_{n-1})$ the subspace generated by vectors $x_1,x_2,...,x_{n-1}$. The following definition of a strong convex n-normed space was introduced in [2]: the n-normed vector space $(L,\|\bullet,...,\bullet\|)$ we call strong convex if

$$\|a+b,x_1,x_2,...,x_{n-1}\|=\|a,x_1,x_2,...,x_{n-1}\|+\|b,x_1,x_2,...,x_{n-1}\|;$$
$$\|a,x_1,x_2,...,x_{n-1}\|=\|b,x_1,x_2,...,x_{n-1}\|=1 \text{ and}$$
$$P(a,b)\cap P(x_1,x_2,...,x_{n-1})=\{0\}$$

implies $a=b$.

Let $X=(-\infty,\infty)$ and μ be the Lebeswue measure on X. The functions

$$a(t)=\begin{cases}0, & t\in(-\infty,0)\\ 1-\frac{1}{2^j}, & t\in[j-1,j), j=1,2,3,4...\end{cases}$$

$$b(t)=\begin{cases}0, & t\in(-\infty,1)\\ 1-\frac{1}{2^j}, & t\in[j,j+1), j=1,2,3,4...\end{cases} \text{ and}$$

$$x_i(t)=\begin{cases}1, & t\in[i-1,i)\\ 0, & t\in(-\infty,+\infty)\setminus[i-1,i)\end{cases}, \quad i=1,2,...,n-1$$

belongs to the space $L^1(\mu)$. It is easy to see that

$$P(a,b)\cap P(x_1,...,x_{n-1})=\{0\},\ \|a,x_1,...,x_{n-1}\|=\|b,x_1,...,x_{n-1}\|=1,$$

$$\|a+b,x_1,x_2,...,x_{n-1}\|=2=\|a,x_1,x_2,...,x_{n-1}\|+\|b,x_1,x_2,...,x_{n-1}\|,$$

but $a \neq b$, which means that $(L^1(\mu), \|\bullet,...,\bullet\|)$ is not strong convex.

3. In [3] was given the following definition of a strong n-convex n-normed space: we call the n-normed space $(L, \|\bullet,...,\bullet\|)$ a strong n-convex if for every vectors $x_1,...,x_{n+1} \in L$ which satisfies the conditions:

$$\|x_1,...,x_{i-1},x_{i+1},...,x_n,x_{n+1}\| = \frac{1}{n+1}\|x_1 + x_{n+1}, x_2 + x_{n+1},...,x_n + x_{n+1}\| = 1, \text{ for } i = 1,2,...,n+1$$

it is true that $x_{n+1} = \sum_{i=1}^{n} x_i$.

In the same work it is proved that every strong convex n-normed space is strong n-convex. The converse is not true. Namely, if $n = 2$ in [5] was given an example of a strong n-convex n-normed space which is not strong convex. It is naturally to ask does $(L^1(\mu), \|\bullet,...,\bullet\|)$ is a strong n-convex n-normed space.

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$L^1(\mu)$ КАКО *n*-НОРМИРАН ПРОСТОР

Резиме

Концептот за *n*-норма на векторски простор со димензија поголема или еднаква на *n*, *n*>1, воведен од A.Misiak ([4]), е повеќедимензионална аналогија на концептот за норма. Во [1], [2], [3] и [4] се докажани повеќе својства на *n*-нормираните простори. Во оваа работа ќе докажеме дека, ако μ е позитивна мера во произволен измерлив простор X и $L^1(\mu)$ е просторот реални измерливи функции f во X, тогаш во $L^1(\mu)$ може да се воведе *n*-норма.