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n-SEMINORMED SPACE

Abstract. *In [7] was introduced the concept of a* n-*normed space as a natural analogy of the concept of the normed space. In this work we will consider the concept of a* n-*seminorm as an analogy of the seminorms on a vector space.*

1. Definition and basic properties of a *n*-seminormed space

Definition 1. Let L be a vector space such that dim $L > n$-1 and $p: L^n \to R$ be a mapping such that

i) If $\{x_1,\dots,x_n\} \subseteq L$ is a lineary dependent set, then $p(x_1,\dots,x_n)=0$.

ii) If $\{x_1,\dots,x_n\} \subseteq L$ and $\pi: \{x_1,\dots,x_n\} \to \{x_1,\dots,x_n\}$ is an arbitrary bejection, then $p(x_1,\dots,x_n) = p(\pi(x_1),\dots,\pi(x_n))$.

iii) If $\{x_1,\dots,x_n\} \subseteq L$ and $\alpha \in R$, then $p(\alpha x_1,\dots,x_n) = |\alpha| \cdot p(x_1,\dots,x_n)$.

iv) For every $y_1, x_1,\dots,x_n \in L$ it is true

$$p(x_1 + y_1, x_2 \dots, x_n) \le p(x_1, x_2 \dots, x_n) + p(y_1, x_2 \dots, x_n).$$

We call the mapping p a ***n*-seminorm** and the pair (L,p) ***n*-seminormed space**.

Note 1. From definition 1 can be concluded that it is possible $p(x_1,\dots,x_n)=0$ when $\{x_1,\dots,x_n\} \subseteq L$ is lineary independent, which is impossible in the case of a *n*-norm. It is clear that every *n*-norm is a *n*-seminorm.

Examples. a) Let l^∞ be the set of all bounded sequences of real numbers. This set with usual operations of adding and product of a sequence with real number is a real vector space, ([8]).

Let j_k, $k = 1,2,\dots,n$ are given natural numbers such that $j_1 < j_2 < \dots < j_n$ and let $x_i = \left(x_{ij}\right)_{j=1}^{\infty} \in l^{\infty}$, $i = 1,2,\dots,n$. With

$$p_{j_1,j_2,\dots,j_n}(x_1,x_2,\dots,x_n) = \left\| \begin{vmatrix} x_{1j_1} & x_{1j_2} & \dots & x_{1j_{n-1}} & x_{1j_n} \\ x_{2j_1} & x_{2j_2} & \dots & x_{2j_{n-1}} & x_{2j_n} \\ \dots & \dots & \dots & \dots & \dots \\ x_{nj_1} & x_{nj_2} & \dots & x_{nj_{n-1}} & x_{nj_n} \end{vmatrix} \right\|,$$

we define a function $p_{j_1,\dots,j_n} : l^{\infty} \text{ x} \dots \text{ x } l^{\infty} \to R$. Since $x_i = \left(x_{ij}\right)_{j=1}^{\infty} \in l^{\infty}$, $i = 1,2,\dots,n$ there exist constants M_i, $i = 1,2,\dots,n$ such that $\left|x_{ij}\right| \le M_i$, for every $j \in N$, from which follows $p_{j_1,j_2,\dots,j_n}(x_1,\dots,x_n) \le n!\prod_{i=1}^{n} M_i$. This means that the function $p_{j_1,\dots,j_n}$ is well defined. From the properties of the determinants and the criterion for the linear dependece in l^{∞} it follows that $p_{j_1,\dots,j_n}$ is a n-seminorm in l^{∞}.

b) Let X be an arbitrary measure space and μ be a positive measure on X. We denote with $L^1(\mu)$ the set of all real measarable functions f on X, such that $\int_X |f| d\mu < \infty$. The set $L^1(\mu)$ with operations of adding of functions and product with real number is a real vector space (th. 1.32, [9]).

Let $n \ge 2$ and $E_1,\dots,E_n$ be are arbitrary measarable subsets of X and $f_1,\dots,f_n \in L^1(\mu)$. With

$$p_{E_1,E_2,\dots,E_n}(f_1,f_2,\dots,f_n) = \left\| \begin{vmatrix} \int_{E_1} f_1 d\mu & \int_{E_2} f_1 d\mu & \dots & \int_{E_{n-1}} f_1 d\mu & \int_{E_n} f_1 d\mu \\ \int_{E_1} f_2 d\mu & \int_{E_2} f_2 d\mu & \dots & \int_{E_{n-1}} f_2 d\mu & \int_{E_n} f_2 d\mu \\ \dots & \dots & \dots & \dots & \dots \\ \int_{E_1} f_n d\mu & \int_{E_2} f_n d\mu & \dots & \int_{E_{n-1}} f_n d\mu & \int_{E_n} f_n d\mu \end{vmatrix} \right\|$$

we define a function $p_{E_1,E_2,\dots,E_n} : L^1(\mu) \text{ x} \dots \text{ x } L^1(\mu) \to R$.. According to theorem 1.33, [9], for every measarable set E it is true

$$\left|\int_E f_i d\mu\right| \le \int_E |f_i| d\mu \le \int_X |f_i| d\mu = M_i < \infty,$$

for $i=1,\ldots,n$, from which follows $p_{E_1,E_2,\ldots,E_n}(f_1,\ldots,f_n)\le n!\prod_{i=1}^{n}M_i$. This means that the function $p_{E_1,E_2,\ldots,E_n}$ is well defined. From the properties of the determinants and the criterion for the linear dependence in $L^1(\mu)$ follows that $p_{E_1,E_2,\ldots,E_n}$ is a n-seminorm on $L^1(\mu)$. ♦

Lemma 1. Let (L,p) be a n-seminormed space. For every $a,x_1,\ldots,x_{n-1}\in L$ and every $\lambda_1,\ldots,\lambda_{n-1}\in R$ it is valid the following identity

$$p(a,x_1,\ldots,x_{n-1})=p\left(a+\sum_{i=1}^{n-1}\lambda_i x_i,x_1,\ldots,x_{n-1}\right). \quad (1)$$

Proof. Trivial. ♦

Lemma 2. Let (L,p) be a n-seminormed space. Then for every $a,b,x_3,\ldots,x_n\in L$ and every $x_1,x_2\in L$ such that $x_1=\alpha a+\beta b$, $x_2=\gamma a+\delta b$, where $\alpha,\beta,\gamma,\delta$ are arbitrary reall numbers, we have

$$p(x_1,x_2,x_3,\ldots,x_n)=|\alpha\delta-\beta\gamma|\cdot p(a,b,x_3,\ldots,x_n).$$

Proof. Trivial. ♦

Theorem 1. Let p be a n-seminorm on the vector space L. Then,

a) $|p(x_1,x_2\ldots,x_n)-p(y_1,x_2\ldots,x_n)|\le p(x_1-y_1,x_2\ldots,x_n)$, for every $y_1,x_1,\ldots,x_n\in L$.

b) $p(x_1,x_2\ldots,x_n)\ge 0$, for every $y_1,x_1,\ldots,x_n\in L$.

c) For every $x_1,\ldots,x_{n-1}\in L$ the set $\{x|\ p(x,x_1,\ldots,x_{n-1})=0\}$ is a subspace of L.

Proof. a) We have,

$$p(x_1,x_2\ldots,x_n)=p(x_1-y_1+y_1,x_2\ldots,x_n)\le p(x_1-y_1,x_2\ldots,x_n)+p(y_1,x_2\ldots,x_n)$$

$$p(x_1,x_2\ldots,x_n)-p(y_1,x_2\ldots,x_n)\le p(x_1-y_1,x_2\ldots,x_n). \quad (2)$$

Simillary:

$$p(y_1,x_2\ldots,x_n)-p(x_1,x_2\ldots,x_n)\le p(y_1-x_1,x_2\ldots,x_n)=p(x_1-y_1,x_2\ldots,x_n). \quad (3)$$

Now, the statement follows from the inequalyties (2) i (3).

b) If $y_1=0$, then from a) and *i)* in definition 1 we have

$$p(x_1,x_2\ldots,x_n)\ge|p(x_1,x_2\ldots,x_n)|\ge 0, \text{ for every } x_1,\ldots,x_n\in L.$$

c) Let $x_1,\ldots,x_{n-1}\in L$, $\alpha,\beta\in R$ and $x,y\in L$ satisfies:

$$p(x,x_1,x_2\dots,x_{n-1})=p(y,x_1,x_2\dots,x_{n-1})=0.$$

Then,

$$0\le p(\alpha x+\beta y,x_1,x_2\dots,x_{n-1})\le p(\alpha x,x_1,x_2\dots,x_{n-1})+p(\beta y,x_1,x_2\dots,x_{n-1})=$$
$$=|\alpha|\cdot p(x,x_1,x_2\dots,x_{n-1})+|\beta|\cdot p(y,x_1,x_2\dots,x_{n-1})=0$$

and so $p(\alpha x+\beta y,x_1,x_2\dots,x_{n-1})=0$, which means that $\{x|\ p(x,x_1,\dots,x_{n-1})=0\}$ is a subspace of L. ♦

Lemma 3. Let $x_1,\dots,x_n\in L$ and p be a n-seminorm in L. Then the function $p_1:L\to R$ defined with $p_1(x)=\sum_{j_k\ne j_i}p(x,x_{j_1},x_{j_2},\dots,x_{j_{n-1}})$, for every $x\in L$, is a seminorm in L.

Proof. For every $x\in L$ and $\alpha\in R$ it is true

$$p_1(\alpha x)=\sum_{j_k\ne j_i}p(\alpha x,x_{j_1},x_{j_2},\dots,x_{j_{n-1}})=\sum_{j_k\ne j_i}|\alpha|p(x,x_{j_1},x_{j_2},\dots,x_{j_{n-1}})$$
$$=|\alpha|\sum_{j_k\ne j_i}p(x,x_{j_1},x_{j_2},\dots,x_{j_{n-1}})=|\alpha|\cdot p_1(x).$$

For every $x,y\in L$ we have:

$$p_1(x+y)=\sum_{j_k\ne j_i}p(x+y,x_{j_1},x_{j_2},\dots,x_{j_{n-1}})\le\sum_{j_k\ne j_i}p(x,x_{j_1},x_{j_2},\dots,x_{j_{n-1}})+p(y,x_{j_1},x_{j_2},\dots,x_{j_{n-1}})$$
$$=\sum_{j_k\ne j_i}p(x,x_{j_1},x_{j_2},\dots,x_{j_{n-1}})+\sum_{j_k\ne j}p(y,x_{j_1},x_{j_2},\dots,x_{j_{n-1}})=p_1(x)+p_1(y).\ ♦$$

2. n-seminorms and n-pseudometrics

The concept of a n-pseudometrics was introduced in [3], as follows:

Definition 2. Let M be a set and $d:M^{n+1}\to R$ be a function such that:

a) For every $x_1,x_2,\dots,x_n,x_{n+1},x_{n+2}\in M$ it is true

$$d(x_1,\dots,x_{n+1})\le d(x_2,\dots,x_{n+1},x_{n+2})+\dots+d(x_{j+1},\dots,x_{n+2},x_1,\dots,x_{j-1})+\dots+d(x_{n+2},x_1,\dots,x_n)$$

b) If $x_{n+2}=x_1$ or $x_{n+2}=x_2$ or ... or $x_{n+2}=x_{n-1}$ or $x_{n+2}=x_n$ then

$$d(x_1,\dots,x_{n+1})=d(x_2,\dots,x_{n+1},x_{n+2})+\dots+d(x_{j+1},\dots,x_{n+2},x_1,\dots,x_{j-1})+\dots+d(x_{n+2},x_1,\dots,x_n)$$

We call the function d a n-pseudometric, and we call the pair (M,d) a n-pseudometric space.

Note 2. To simplicity the denotation in this section the vallue $d(x_1,\dots,x_{n+1})$ will be denoted by X_o or Y_o, the vallue $d(x_2,\dots,x_{n+1},x_{n+2})$ will be denoted by X_1 or Y_1, with X_j will be denoted the vallue $d(x_{j+1},...,x_{n+2},x_1,...,x_{j-1})$, and Y_j is the vallue of $d(x_1,...,x_{j-1},x_{j+1},...,x_{n+2})$, for $2 \leq j \leq n+1$.

In the next theorems will be presented the basic results of the n-pseudometrics and the n-psevdometrics spaces which are proved in [3].

Theorem 2. Let (M,d) be a n-pseudometric space. If mostly m, of the points $x_1,x_2,...,x_n,x_{n+1} \in M$, $2 \leq m \leq n+1$ are identical, then $X_o = 0$. ♦

Theorem 3. Let $x_1,x_2,...,x_n,x_{n+1} \in M$. If d is a n-pseudometric, then

$$d(x_1,x_2,...,x_n,x_{n+1}) = d(\pi(x_1),\pi(x_2),...,\pi(x_n),\pi(x_{n+1})) \tag{4}$$

for every bejection $\pi:\{x_1,...,x_{n+1}\} \to \{x_1,...,x_{n+1}\}$. ♦

Corollary 1. If (M,d) is a n-pseudometric space, then, (the notations are as in Note 2), for every $x_1,...,x_{n+1},x_{n+2} \in M$ it is true $Y_0 \leq Y_1 + Y_2 + ... + Y_{n+1}$. ♦

Theorem 4. If (M,d) is a n-pseudometric space, then for every $z_1,...,z_{n+1} \in M$ it is true $d(z_1,z_2,...,z_{n+1}) \geq 0$. ♦

Theorem 5. Let (M,d) be a n-pseudometric space. Then, (the notations are as in Note 2), for every $x_1,x_2,...,x_{n+1},x_{n+2} \in M$, from $Y_2 = Y_3 = Y_4 = ... = Y_{n+1} = 0$ it follows $Y_1 = Y_0$. ♦

In the next theorem we will prove that every n-seminorm define a n-pseudometric.

Theorem 6. Let (L,p) be a n-seminormed space and let $x_1,x_2,...,x_{n+1} \in L$. Then the function $d: L^{n+1} \to R$, defined with

$$d(x_1,x_2,...,x_{n+1}) = p(x_1 - x_{n+1}, x_2 - x_{n+1},...,x_n - x_{n+1}). \tag{5}$$

is a n-pseudometric, which means that (L,d) is a n-pseudometric space.

Proof. Let $x_1,x_2,...,x_n,x_{n+1},x_{n+2} \in L$. From the definition of a n-seminormed space and Lemma 1 it follows:

$$\begin{aligned}
d(x_1,x_2,...,x_{n+1}) &= p(x_1 - x_{n+1}, x_2 - x_{n+1},...,x_n - x_{n+1}) \\
&= p(x_1 - x_{n+2} + x_{n+2} - x_{n+1}, x_2 - x_{n+1},...,x_n - x_{n+1}) \\
&\leq p(x_1 - x_{n+2}, x_2 - x_{n+1},...,x_n - x_{n+1}) + p(x_{n+2} - x_{n+1}, x_2 - x_{n+1},...,x_n - x_{n+1}) \\
&\leq p(x_{n+2} - x_{n+1}, x_2 - x_{n+1},...,x_n - x_{n+1}) + p(x_1 - x_{n+2}, x_2 - x_{n+1},...,x_n - x_{n+1}) \\
&= p(x_{n+2} - x_{n+1}, x_2 - x_{n+2},...,x_n - x_{n+2}) + p(x_1 - x_{n+2}, x_2 - x_{n+2} + x_{n+2} - x_{n+1},...,x_n - x_{n+1}) \\
&\leq d(x_2,x_3,...,x_{n+2}) + p(x_1 - x_{n+2}, x_{n+2} - x_{n+1},...,x_n - x_{n+1}) + p(x_1 - x_{n+2}, x_2 - x_{n+2},...,x_n - x_{n+1}) \\
&= d(x_2,x_3,...,x_{n+2}) + p(x_1 - x_{n+2}, x_{n+2} - x_{n+1},...,x_n - x_{n+2}) + p(x_1 - x_{n+2}, x_2 - x_{n+2},...,x_n - x_{n+1})
\end{aligned}$$

$$= d(x_2, x_3, \ldots, x_{n+2}) + d(x_1, x_3, \ldots, x_{n+2}) + p(x_1 - x_{n+2}, x_2 - x_{n+2}, \ldots, x_n - x_{n+1})$$
$$\leq \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\leq d(x_2, x_3, \ldots, x_{n+2}) + d(x_1, x_3, \ldots, x_{n+2}) + \ldots + d(x_1, x_2, \ldots, x_{n-1}, x_{n+1}, x_{n+2}) + d(x_1, x_2, \ldots, x_n, x_{n+2})$$

and so the property a) of the definition 2 it is true.

If $x_{n+2} = x_i$, for some $i \in \{1,2,\ldots,n+1\}$ then for $i \neq j$ it is true

$$d(x_{j+1}, \ldots, x_{n+2}, x_1, \ldots, x_{j-1}) = p(x_{j+1} - x_{j-1}, \ldots, x_{n+2} - x_{j-1}, x_1 - x_{j-1}, \ldots, x_{j-2} - x_{j-1})$$
$$= p(x_1 - x_{j-1}, \ldots, x_i - x_{j-1}, \ldots, x_{j-2} - x_{j-1}, x_{j+1} - x_{j-1}, \ldots, x_{n+2} - x_{j-1})$$
$$= p(x_1 - x_{j-1}, \ldots, x_i - x_{n+2}, \ldots, x_{j-2} - x_{j-1}, x_{j+1} - x_{j-1}, \ldots, x_{n+2} - x_{j-1})$$
$$= p(x_1 - x_{j-1}, \ldots, 0, \ldots, x_{j-2} - x_{j-1}, x_{j+1} - x_{j-1}, \ldots, x_{n+2} - x_{j-1}) = 0,$$

and if $i = j$, then

$$d(x_{j+1}, \ldots, x_{n+1}, x_j, x_1, \ldots, x_{j-1}) =$$
$$= p(x_{j+1} - x_{j-1}, \ldots, x_{n+1} - x_{j-1}, x_j - x_{j-1}, x_1 - x_{j-1}, \ldots, x_{j-2} - x_{j-1})$$
$$= p(x_1 - x_{n+1}, \ldots, x_j - x_{n+1}, x_{j+1} - x_{n+11}, \ldots, x_n - x_{n+1})$$
$$= d(x_1, \ldots, x_j, x_{j+1}, \ldots, x_{n+1}),$$

and so

$$d(x_1, \ldots, x_{n+1}) = d(x_2, \ldots, x_{n+1}, x_{n+2}) + \ldots + d(x_{j+1}, \ldots, x_{n+2}, x_1, \ldots, x_{j-1}) + \ldots + d(x_{n+2}, x_1, \ldots, x_n)$$

This means that the property b) of the definition 2 is true. ♦

3. *n*-norms and *n*-metrics

At the end of this work we will consider the concept of the n-metric space considering in [3] and [7] and the connection between n-metrics and n-norms.

Definition 3. A n-pseudometric space (M,d) we call n-metric iff $|M| \geq n+1$ and for every different elements $x_1, x_2 \in M$ there is a set of $n-1$ points $\{x_3, \ldots, x_{n+1}\} \subseteq M$, such that $d(x_1, x_2, \ldots, x_{n+1}) \neq 0$.

Theorem 7. Let $(L, \|\bullet, \ldots, \bullet\|)$ be a n-normed space and let $x_1, x_2, \ldots, x_{n+1} \in L$. Then the function $d: L^{n+1} \to R$, defined by

$$d(x_1, x_2, \ldots, x_{n+1}) = \|x_1 - x_{n+1}, x_2 - x_{n+1}, \ldots, x_n - x_{n+1}\| \quad (6)$$

is a n-metric, whivh means that (L,d) is a n-metric space.

Proof. We say in Note 1 that every n-norm is a n-seminorm. According th. 6 with (6) is defined a n-pseudometric and so $(L,\|\bullet,\ldots,\bullet\|)$ is a n-pseudometric space.

Let x_1, x_2 be a different elements of L. Without lousing the generallity we may assume that $x_2 \neq 0$. But, L is a n-dimensional vectors space, and so there exist $x_3, x_4 \ldots, x_{n+1} \in L$ such that $\{x_2 - x_1, x_3 - x_1, \ldots, x_{n+1} - x_1\}$ is a lineary independent set, which means

$$d(x_1, x_2, x_3, x_4, \ldots, x_{n+1}) = \|x_1 - x_{n+1}, x_2 - x_{n+1}, \ldots, x_n - x_{n+1}\| =$$
$$= \|x_2 - x_1, x_3 - x_1, x_4 - x_1, \ldots, x_{n+1} - x_1\| \neq 0$$

and so $(L,\|\bullet,\ldots,\bullet\|)$ is a n-metric space. ♦

Definition 4. We will say that the n-metric d is a translatary invariant if for every $x_1, x_2, \ldots, x_{n+1} \in L$ and every $a \in L$ it is true

$$d(x_1 + a, x_2 + a, \ldots, x_{n+1} + a) = d(x_1, x_2, \ldots, x_{n+1}).$$

Lemma 4. If $(L,\|\bullet,\ldots,\bullet\|)$ is a n-normed space in which with (6) is defined a n-metric d, then

a) d is translatary invariant,

b) For every $x_1, x_2, \ldots, x_{n+1} \in L$ and every $y = x_2 + \alpha(x_1 - x_2)$, $\alpha \in [0,1]$ it is true

$$d(x_1, y, x_3, \ldots, x_{n+1}) = (1-\alpha)d(x_1, x_2, \ldots, x_{n+1}),\ d(y, x_2, x_3, \ldots, x_{n+1}) = \alpha d(x_1, x_2, \ldots, x_{n+1}),$$
$$d(x_1, x_2, \ldots, x_{n+1}) = d(y, x_2, x_3, \ldots, x_{n+1}) + d(x_1, y, x_3, \ldots, x_{n+1}).$$

c) For every $x_1, x_2, \ldots, x_{n+1} \in L$ and

$$y = \sum_{i=1}^{n+1} \alpha_i x_i,\ \alpha_i \in [0,1],\ i = 1,2,\ldots,n+1 \text{ and } \sum_{i=1}^{n+1} \alpha_i = 1$$

it is true

$$d(x_1, x_2, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_{n+1}) = \alpha_i d(x_1, x_2, \ldots, x_{n+1}) \text{ and}$$
$$d(x_1, x_2, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} d(x_1, x_2, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_{n+1}).$$

Proof. a) For every $x_1, x_2, \ldots, x_{n+1} \in L$ and every $a \in L$ it's true

$$d(x_1 + a, x_2 + a, \ldots, x_{n+1} + a) =$$
$$= \|x_1 + a - (x_{n+1} + a), x_2 + a - (x_{n+1} + a), \ldots, x_n + a - (x_{n+1} + a)\|$$
$$= \|x_1 - x_{n+1}, x_2 - x_{n+1}, \ldots, x_n - x_{n+1}\| = d(x_1, x_2, \ldots, x_{n+1}),$$

which means that d is a translatary invariant n-metric.

b) Let $x_1, x_2, \ldots, x_{n+1} \in L$ and $y = x_2 + \alpha(x_1 - x_2)$, $\alpha \in [0,1]$. Then

$$\begin{aligned} d(x_1, y, x_3, \ldots, x_{n+1}) &= \|y - x_1, x_3 - x_1, \ldots, x_{n+1} - x_1\| \\ &= \|x_2 - x_1 - \alpha(x_2 - x_1), x_3 - x_1, \ldots, x_{n+1} - x_1\| \\ &= (1-\alpha)\|x_2 - x_1, x_3 - x_1, \ldots, x_{n+1} - x_1\| = (1-\alpha) d(x_1, x_2, \ldots, x_{n+1}), \end{aligned}$$

and

$$\begin{aligned} d(y, x_2, x_3, \ldots, x_{n+1}) &= \|y - x_2, x_3 - x_2, \ldots, x_{n+1} - x_2\| \\ &= \|x_2 + \alpha(x_1 - x_2) - x_2, x_3 - x_2, \ldots, x_{n+1} - x_2\| \\ &= \alpha\|x_1 - x_2, x_3 - x_2, \ldots, x_{n+1} - x_2\| = \alpha d(x_1, x_2, \ldots, x_{n+1}). \end{aligned}$$

Hence

$$\begin{aligned} d(x_1, x_2, \ldots, x_{n+1}) &= (1-\alpha) d(x_1, x_2, \ldots, x_{n+1}) + \alpha d(x_1, x_2, \ldots, x_{n+1}) \\ &= d(y, x_2, x_3, \ldots, x_{n+1}) + d(x_1, y, x_3, \ldots, x_{n+1}). \end{aligned}$$

c) Let $x_1, x_2, \ldots, x_{n+1} \in L$ and $y = \sum_{i=1}^{n+1} \alpha_i x_i$, $\alpha_i \in [0,1]$, $i = 1,2,\ldots,n+1$, $\sum_{i=1}^{n+1} \alpha_i = 1$.

Then

$$\begin{aligned} d(x_1, x_2, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_{n+1}) &= \|x_2 - x_1, \ldots, x_{i-1} - x_1, y - x_1, x_{i+1} - x_1, x_{n+1} - x_1\| \\ &= \left\|x_2 - x_1, \ldots, x_{i-1} - x_1, y - x_1 - \sum_{j \neq i} \alpha_j (x_j - x_1), x_{i+1} - x_1, x_{n+1} - x_1\right\| \\ &= \|x_2 - x_1, \ldots, x_{i-1} - x_1, \alpha_i(x_i - x_1), x_{i+1} - x_1, x_{n+1} - x_1\| \\ &= \alpha_i \|x_2 - x_1, \ldots, x_{i-1} - x_1, x_i - x_1, x_{i+1} - x_1, x_{n+1} - x_1\| = \alpha_i d(x_1, x_2, \ldots, x_{n+1}). \end{aligned}$$

According to the above

$$\begin{aligned} d(x_1, x_2, \ldots, x_{n+1}) &= d(x_1, x_2, \ldots, x_{n+1}) \sum_{i=1}^{n+1} \alpha_i = \sum_{i=1}^{n+1} \alpha_i d(x_1, x_2, \ldots, x_{n+1}) \\ &= \sum_{i=1}^{n+1} d(x_1, x_2, \ldots, x_{i-1}, y, x_{i+1, \ldots} x_{n+1}). \ \blacklozenge \end{aligned}$$

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n-ПОЛУНОРМИРАН ПРОСТОР

Резиме

Во [7] е воведен поимот *n*-нормиран простор, како природна аналогија на поимот нормиран простор. Во оваа работа ќе го разгледаме поимот *n*-полунорма, како аналогија на полунормите во векторски простор.