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SOME RELATIONS BETWEEN BESSEL AND BERNOULLI POLYNOMIALS

An expansion of Bessel polynomials through Bernoulli polynomials is given.

1. The simple Bessel polynomials

$$y_n(x) = {}_2F_0\left(-n, 1+n; -; -\frac{x}{2}\right)$$

and the generalized one

$$y_n(a,b;x) = {}_2F_0\left(-n, a-1+n; -; -\frac{x}{b}\right)$$

may be considered as special cases of the polynomials

$$\varphi_n(c;x) = \frac{(c)_n}{n!}\,{}_2F_0(-n, c+n; -; x)$$

Really, we have $\varphi_n\left(1, -\frac{x}{2}\right) = y_n(x)$

and $\frac{n!}{(a-1)_n}\varphi_n\left(a-1, -\frac{x}{b}\right) = y(a,b;x)$

respectively [1].

${}_2F_1(\alpha,\beta;\gamma;x)$ is hypergeometric function defined by

$${}_2F_1(\alpha,\beta;\gamma;x) = \sum_{n=1}^{\infty}\frac{(\alpha)_n(\beta)_n x^n}{(\gamma)_n n!}$$

with
$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1), n \geq 1$$
$$(\alpha)_0 = 1, \alpha \neq 0$$

the Pochhammer symbol. Similarly, ${}_pF_q(\alpha_1,\alpha_2,...,\alpha_p;\beta_1,\beta_2,...,\beta_q;x)$ is generalized hypergeometric function used later.

The Bernoulli polynomials $B_n(x)$ are defined by [2]

$$\frac{te^{tx}}{e^t-1}=\sum_{k=0}^{n} B_n(x)\frac{t^n}{n!} \tag{1}$$

Two of the properties of Bernoulli polynomial arising from (1) are

$$B_n(x+1)-B_n(x)=nx^{n-1}, n=0,1,2,... \tag{2}$$

$$B_n(x+1)=\sum_{k=0}^{n}\binom{n}{k}B_k(x) \tag{3}$$

From (2) and (3) we find

$$x^n=\sum_{k=0}^{n}\frac{n!B_k(x)}{k!(n-k+1)!} \tag{4}$$

the expansion (4) is used bellow in the method of technique applied for expanding the polynomials.

The purpose of this paper is to give the expansions of the polynomial $\varphi_n(x,c)$ and the Bessel polynomials $y_n(x)$ and $y_n(a,b,x)$ in series of Bernoulli polynomials.

2. Consider the series

$$\begin{aligned}\sum_{k=0}^{\infty}\varphi_n(c,x)t^n &=\sum_{k=0}^{\infty}\sum_{s=0}^{n}\frac{(-1)^s(c)_{n+s}x^s t^n}{s!(n-s)!}\\ &=\sum_{n,s=0}^{\infty}\frac{(-1)^s(c)_{n+2s}x^s t^{n+s}}{s!n!}\\ &=\sum_{n,s=0}^{\infty}\sum_{k=0}^{s}\frac{(-1)^s(c)_{n+2s}B_k(x)t^{n+s}}{n!(s-k+1)!}\\ &=\sum_{n,k,s=0}^{\infty}\frac{(-1)^{s+k}(c)_{n+2s+2k}B_k(x)t^{n+s+k}}{n!k!(s+1)!}\end{aligned}$$

in which we have used the identity

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A(n,k) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+k) \tag{5}$$

to collect the powers of t in the last summation above.
Using the same identity (5) in reverse, we may write

$$\sum_{n=0}^{\infty}\varphi_n(c,x)t^n = \sum_{n,k=0}^{\infty}\sum_{s=0}^{n}\frac{(-1)^{s+k}(c)_{n+s+2k}B_k(x)t^{n+k}}{(s+1)!(n-s)!k!}$$

$$= \sum_{n,k=0}^{\infty}\sum_{s=0}^{n}\frac{(-1)^k(c+n+2k)_s(-n)_s(1)_s(c)_{n+2k}B_k(x)t^{n+k}}{s!(2)_s n!k!}$$

$$= \sum_{n,k=0}^{\infty} {}_3F_1(-n,c+n+2k,1;2;1)\frac{(-1)^k(c)_{n+2k}B_k(x)t^{n+k}}{n!k!}$$

Applying the identity (5) again, we obtain

$$\sum_{n=0}^{\infty}\varphi_n(c,x)t^n =$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} {}_3F_1(-n+k,c+n+k,1;2;1)\frac{(-1)^k(c)_{n+k}B_k(x)t^n}{k!(n-k)!}$$

Finally we have

$$\varphi_n(c,x) = \sum_{k=0}^{n} {}_3F_1(-n+k,c+n+k,1;2;1)\frac{(-1)^k(c)_{n+k}B_k(x)}{k!(n-k)!}$$

3. For the Bessel polynomials we find

$$y_n(x) = \sum_{k=0}^{n} {}_3F_1(-n+k,1+n+k,1;2;1)\frac{(-1)^k(n+k)!B_k\left(-\frac{k}{2}\right)}{k!(n-k)!} \text{ or}$$

$$y_n(x) = \sum_{k=0}^{n} {}_3F_1\left(-n+k,1+n+k,1;2;-\frac{1}{2}\right)\frac{(n+k)!B_k(x)}{k!(n-k)!2^k}$$

and

$$y_n(a,b;x) = \sum_{k=0}^{n} {}_3F_1(-n+k, a-1+k+n, 1; 2; 1)(-1)^k \binom{n}{k} (a-1+n)_k B_k\left(-\frac{x}{b}\right)$$

or

$$y_n(a,b;x) = \sum_{k=0}^{n} {}_3F_1\left(-n+k, a-1+k+n, 1; 2; -\frac{1}{b}\right) \binom{n}{k} \frac{(a-1+n)_k}{b^k} B_k(x)$$

References

1. E.Rainville, Special functions, The Macmillan Company, New York, 1960
2. A.Erdelyi et al. Higher Transcedental Functions, Vol.1, McGraw-Hill, New York-Toronto-London, 1952