Математички Билтен ISSN 0351-336X (print)
37(LXIII) No. 1 ISSN 1857-9914 (online)
2013(61-74) UDC: 519.173.174
Скопје, Македонија

WEAK-ODD EDGE-COLORING OF DIGRAPHS

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Abstract. A weak-odd edge-coloring of a digraph D is a (not necessarily proper) edge-coloring such that for each vertex $v \in V(D)$ at least one color c satisfies the following requirement: if $d^+(v) > 0$ then c appears an odd number of times on the outgoing edges at v; and if $d^-(v) > 0$ then c appears an odd number of times on the ingoing edges at v. The minimum number of colors sufficient for a weak-odd edge-coloring of D is the weak-odd chromatic index, denoted $\chi'_{\mathrm{wo}}(D)$. In this article we prove that $\chi'_{\mathrm{wo}}(D) \leq 3$ for every digraph D, and show that this bound is sharp. We study when does a graph admit an orientation so that the obtained digraph is weak-odd 1-edge-colorable. We also prove that every graph admits an orientation for which the obtained digraph is weak-odd 2-edge-colorable.

1. Introduction

1.1. **Terminology and notation.** Throughout the article we mainly follow terminology and notation used in [1, 7]. All considered digraphs and graphs are finite (i.e. have finite sets of vertices and edges). A *directed graph* (or *digraph*) D is a triple consisting of a vertex set $V(D)$, an edge set $E(D)$, and a map which assigns to each edge an ordered pair of vertices: the first vertex of the ordered pair is the *tail* of the edge, and the second is the *head*; together they are the *endpoints*. Thus each edge is said to be *directed* from its tail to its head. An edge with tail u and head v is referred to as a *directed uv-edge* (the adjective 'directed' is often omitted). Given a vertex $v \in V(D)$, an *outgoing* (resp. *ingoing*) edge at v is any edge having tail (resp. head) v. Denote by $E^+_D(v)$ (resp. $E^-_D(v)$) the set of outgoing (resp. ingoing) edges at v. A *loop* is an edge of a digraph whose endpoints are equal. Each loop at a vertex v belongs to both $E^+_D(v)$ and $E^-_D(v)$. *Parallel edges* are edges having the same ordered pair of endpoints. The size of the set $E^+_D(v)$ (resp. $E^-_D(v)$) is called the *outdegree* $d^+_D(v)$ (resp. *indegree* $d^-_D(v)$) of the vertex v. If $d^+_D(v) = 0$ (resp. $d^-_D(v) = 0$), then v is called a

2000 Mathematics Subject Classification. 05C10.

Key words and phrases. digraph, weak-odd edge-coloring, weak-odd chromatic index.

sink (resp. *source*) of D. Any sink or source is a *peripheral vertex* of D. On the other hand, any vertex that is neither a source nor a sink is an *intermediate vertex* of D. For $v \in V(D)$, the union $E_D(v) = E_D^+(v) \cup E_D^-(v)$ is the set of edges incident to v, and the sum $d_D(v) = d_D^+(v) + d_D^-(v)$ is the *degree* of v. Whenever $d_D(v) = 1$ we say that v is a *pendant vertex*, and then its only incident edge is a *pendant edge* of D. For $X \subseteq V(D) \cup E(D)$, denote by $D - X$ the subdigraph of D obtained by removing X. The *underlying graph* of a digraph D is the graph G acquired by 'forgeting' the direction of each edge, i.e. by treating the edges as unordered pairs: the vertex set and the edge set remain the same, the endpoints of every edge are the same in G as in D, but in G they become an unordered pair. We say that a digraph D is *connected* if its underlying graph is connected.

Similar to digraphs, for a given graph G and a vertex $v \in V(G)$, the subset of $E(G)$ consisting of the edges incident to v is denoted by $E_G(v)$. For every $v \in V(G)$, the size of the set $E_G(v)$ is the *degree* $d_G(v)$ of v (recall that every loop at v is counted twice for $|E_G(v)|$). We refer to each vertex $v \in V(G)$ of even (resp. odd) degree $d_G(v)$ as an *even* (resp. *odd*) vertex of G. A graph is called *even* (resp. *odd*) whenever all its vertices are even (resp. odd).

1.2. **Definition of weak-odd edge-coloring.** Motivated by [3, 4, 6], the following notion of weak-odd edge-coloring of graphs and the related weak-odd chromatic index were considered in [5]. A(not necessarily proper) edge-coloring of a graph G is a *weak-odd edge-coloring* of G if each non-isolated vertex is an odd vertex in at least one of the subgraphs induced by the different color classes. In other words, for each non-isolated vertex $v \in V(G)$, at least one color c appears an odd number of times on $E_G(v)$. An obvious necessary and sufficient condition for weak-odd edge-colorability of G is the absence of vertices incident only to loops. For any such graph G, a weak-odd edge-coloring using at most k colors is referred to as a *weak-odd k-edge-coloring* of G, and we then say that G is *weak-odd k-edge-colorable*. The *weak-odd chromatic index* $\chi'_{\mathrm{wo}}(G)$ is defined as the least integer k for which G is weak-odd k-edge-colorable. The following characterization of G in terms of the value of $\chi'_{\mathrm{wo}}(G)$ was given in [5].

Theorem 1. *For any connected graph G whose edge set does not consist only of loops, it holds that*

$$\chi'_{\mathrm{wo}}(G) = \begin{cases} 0 & \textit{if } G \textit{ is trivial}, \\ 1 & \textit{if } G \textit{ is odd}, \\ 3 & \textit{if } G \textit{ is non-trivial even of odd order}, \\ 2 & \textit{otherwise}. \end{cases}$$

The purpose of this article is to introduce and study an analogous notion for digraphs.

Definition 1. *A (not necessarily proper) edge-coloring of a digraph D is said to be a weak-odd edge-coloring if for each non-isolated vertex $v \in V(D)$ at least one color c is odd at v, i.e. satisfies the following condition:*

(C) if $d_D^+(v) > 0$ then c appears an odd number of times on the outgoing edges at v; and if $d_D^-(v) > 0$ then c appears an odd number of times on the ingoing edges at v.

Thus, in the particular case when v is a peripheral vertex, the condition (C) amounts to the appearance of c an odd number of times on $E_D(v)$.

The minimum number of colors sufficient for a weak-odd edge-coloring of D is the weak-odd chromatic index, denoted $\chi'_{\mathrm{wo}}(D)$. A weak-odd edge-coloring of D using at most k colors is referred to as a *weak-odd k-edge-coloring*, and then D is said to be *weak-odd k-edge-colorable.* Hence, $\chi'_{\mathrm{wo}}(D)$ is the minimum integer k for which D is weak-odd k-edge-colorable.

Remark 1. *We could have defined the notion weak-odd edge-coloring of a digraph D alternatively – by slightly relaxing the condition for each non-isolated vertex $v \in V(D)$:*

(C^*) if $d_D^+(v) > 0$ then at least one color c' appears an odd number of times on $E_D^+(v)$; and if $d_D^-(v) > 0$ then at least one color c'' (not necessarily the same as c') appears an odd number of times on $E_D^-(v)$.

But then, this 'alternative' weak-odd edge-coloring of digraphs would be just a 'disguise' of the notion weak-odd edge-coloring of bipartite graphs with equally sized partite sets. Namely, recall that a split (or bipartization) of a digraph D is a bipartite graph G whose partite sets V^+, V^- are copies of $V(D)$. For each $v \in V(D)$, there is one vertex $v^+ \in V^+$ and one $v^- \in V^-$. For each directed uv-edge in D, there is an edge with endpoints u^+ and v^- in G. Hence, the degrees of the vertices v^+, v^- in the split of D are precisely the outdegree and indegree of v in D, respectively. Furthermore, any bipartite graph G with equally sized partite sets is a split of some digraph D, i.e. can be 'transformed' into D by reversing the described procedure. Therefore, an 'alternative' weak-odd k-edge-coloring of a digraph D is exactly the same thing as a weak-odd k-edge-coloring of its split G. This is the reason why we proceed to study the originally defined notion of weak-odd edge-coloring of digraphs.

In the next section, we prove that every digraph D is weak-odd 3-edge-colorable. Moreover, we provide an example demonstrating that the upper bound $\chi'_{\mathrm{wo}}(D) \leq 3$ is sharp. The following section studies a related problem about orienting graphs. We show that every graph G admits an orientation for which the obtained digraph D is weak-odd 2-edge-colorable. We also solve the decision problem whether a given graph G admits an orientation for which the obtained digraph D is weak-odd 1-edge-colorable.

2. Existence and a tight upper bound for $\chi'_{\mathrm{wo}}(D)$

It is easy to characterize the weak-odd 1-edge-colorable digraphs D. Namely, $\chi'_{\mathrm{wo}}(D) \leq 1$ holds if and only if for every $v \in V(D)$ both $d^+_D(v), d^-_D(v)$ are odd or zero. In particular, $\chi'_{\mathrm{wo}}(D) = 0$ holds precisely when D is empty (i.e. $E(D) = \emptyset$). Next, we give a sufficient condition for weak-odd 2-edge-colorability of digraphs.

Lemma 1. *Let D be a digraph whose underlying graph is a forest. Then, D is weak-odd 2-edge-colorable.*

Proof. We may assume that the underlying graph of D is a nonempty tree. For an arbitrary $v \in V(D)$, consider the set $E_D(v)$. Even if one edge $e \in E_D(v)$ is already colored, say by a color $c \in \{1, 2\}$, this partial coloring of $E_D(v)$ can be turned into a (complete) coloring of $E_D(v)$ using the color set $\{1, 2\}$ such that the color c is odd at v. We describe how this can be done if $e \in E^+_D(v)$, and an analogous coloring procedure works for $e \in E^-_D(v)$. Since e is already colored by c, color the rest of $E^+_D(v)$ by the other color from $\{1, 2\}$. In case v is a source, we are done; otherwise, select an edge $f \in E^-_D(v)$, color it by c, and then color the rest of $E^-_D(v)$ by the other color.

With this coloring procedure in mind, select an arbitrary vertex $v_o \in V(D)$ and color $E_D(v_o)$ using colors from the set $\{1, 2\}$ so that 1 is odd at v_o. Since the underlying graph of D is connected and acyclic, as long as $E(D)$ is not fully colored, there exists a non-pendant vertex $v \in V(D)$ with just one incident edge colored so far, say by a color c. Apply the coloring procedure to $E_D(v)$ so that c ends up being odd at v. By repeating this, we eventually construct a weak-odd 2-edge-coloring of D. □

Clearly, the stated sufficient condition is not necessary for weak-odd 2-edge-colorability (for example, any directed cycle C has $\chi'_{\mathrm{wo}}(C) = 1$). On the other hand, not every digraph is weak-odd 2-edge-colorable (see Fig. 1). We proceed by proving that 3 colors suffice for a weak-odd edge-coloring of every digraph D. Actually, we'll show a slightly stronger result which is more amenable to induction. Recall that every vertex v of a digraph D has precisely one of the following two *types*: either v is an intermediate vertex (i.e. both $d^+_D(v), d^-_D(v) > 0$), or v is a peripheral vertex (a source or sink).

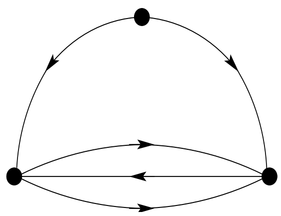

FIGURE 1. A digraph requiring 3 colors for a weak-odd edge-coloring.

Theorem 2. *Every digraph admits a weak-odd edge-coloring using the color set* $\{1, 2, 3\}$ *such that at every peripheral vertex at least one of the colors* 1 *and* 2 *is odd.*

Proof. For simplicity of presentation, just during this proof, we'll refer to any edge-coloring complying the statement of 2 as a *good edge-coloring* of the respective digraph. Suppose the theorem is false, and let D be a counter-example which minimizes $|E(D)|$. Clearly, the digraph D is connected, and we proceed to demonstrate several constraints which apply to it.

Claim 1. *D is without directed cycles, i.e. D is a directed acyclic graph.*

For the sake of contradiction, suppose there exists a directed cycle C in D. Color the edges of C by 3. Consider its edge-complement $\hat{C} = D - E(C)$. We modify the digraph $\hat{C}$ by splitting into $d_{\hat{C}}(v)$ pendant vertices every $v \in V(C)$ of degree $d_{\hat{C}}(v) \geq 2$. After this is done, let D' be the digraph obtained from $\hat{C}$. Since D' is of smaller size than D, there exists a good edge-coloring φ' of D'. But, it is easily observed that we can combine φ' with the already given edge-coloring of C to obtain a good edge-coloring φ of D. The existence of φ contradicts the choice of D, hence establishes the claim.

In particular, we conclude the absence of loops in the digraph D.

Claim 2. *For every edge $e \in E(D)$ there is an endpoint having the different type in $D - e$ and D.*

For the sake of contradiction, suppose there exists such a vw-edge e. We may take a good edge-coloring φ' of $D - e$. We intend to extend φ' to a good edge-coloring φ of D. Let us observe straightaway that if at least one of the vertices v, w is isolated in $D - e$, then there exists a color $c \in \{1, 2\}$ such that $\varphi(e) = c$ fulfils the intention. Hence, we may assume that both

v and w are non-isolated in $D-e$. Delete two colors from the set $\{1,2,3\}$ so that for each of the vertices v and w, at least one of the deleted colors is odd at the that vertex. Say c is the remaining color, and obtain φ by setting $\varphi(e)=c$. Since each of the endpoints of e has the same type in D as in $D-e$, it must be that φ is a good edge-coloring of D, a contradiction which proves the claim.

In particular, there are no (directed) parallel edges in D. Next we show that D is without vertices of outdegree and indegree less than 2.

Claim 3. *There is no vertex $v \in V(D)$ such that both $d_D^+(v), d_D^-(v) \leq 1$.*

Suppose the opposite, i.e. let a vertex v have $d_D^+(v), d_D^-(v) \leq 1$. Assume first that v is an intermediate vertex, and say $e_1 = v_1v$ and $e_2 = vv_2$ are the only two directed edges incident to v. Suppress v, i.e. remove it and add one directed v_1v_2-edge e. The obtained digraph D' is of smaller size than D, hence it admits a good edge-coloring φ'. Construct an edge-coloring φ of D by making it agree with φ' on $E(D)\backslash\{e_1,e_2\}$ and setting $\varphi(e_1) = \varphi(e_2) = \varphi'(e)$. Clearly, φ is a good edge-coloring of D, a contradiction.

Assume now that v is a pendant vertex, and let e be its incident pendant edge. Denote by w the other endpoint of e and consider a good edge-coloring φ' of the smaller digraph $D-v$. Claim 2 implies that w is a peripheral vertex of $D-e$, but an intermediate vertex of D. Hence, at least one color $c \in \{1,2\}$ is odd at w under φ', and we extend φ' to a good edge-coloring φ of D by setting $\varphi(e)=c$. This contradiction proves the claim.

Claim 4. *Any $v \in V(D)$ is a peripheral vertex of D.*

Suppose the opposite, say v is an intermediate vertex of D, i.e. both $d_D^+(v), d_D^-(v)$ are greater than 0. From Claim 3, at least one of $d_D^+(v), d_D^-(v)$ is greater than 1, say $d_D^+(v) \geq 2$. Select an $e \in E_D^+(v)$ and let v_1 be the head of e. Since v is an intermediate vertex of $D-e$, Claim 2 implies that $d_D^-(v_1) = 1$. Hence, Claim 3 yields $d_D^+(v_1) \geq 2$ and we may repeat the argument. Thus, by Claim 1, for any non-negative integer n there exists a directed path $P_n : vv_1 \dots v_n$ in D. But this contradicts the finiteness of D, and establishes the claim.

We are now able to complete the proof. Let T be a spanning tree for the underlying graph of D. By Lemma 1, there exists a weak-odd 2-edge-coloring φ' of the spanning subdigraph $D' = D[E(T)]$ using the color set $\{1,2\}$. Since Claim 4 assures that every vertex of D is peripheral, we can extend φ' to a good edge-coloring φ of D simply by using the color 3 for every edge in $E(D)\backslash E(D')$. But this φ contradicts the choice of D, and thus proves the theorem. □

Corollary 1. *Every digraph D is weak-odd 3-edge-colorable.*

Therefore, every digraph D with at least one vertex v of outdegree $d_D^+(v)$ or indegree $d_D^-(v)$ even and greater than zero, satisfies $2 \le \chi'_{\text{wo}}(D) \le 3$. We believe that a descriptive characterization of digraphs D in terms of the value of $\chi'_{\text{wo}}(D)$ (similar to the one given for graphs in Theorem 1) is impossible. Moreover, we believe that deciding the exact value of $\chi'_{\text{wo}}(D)$ is NP-hard. Henceforth, we turn to another type of problem.

3. A related problem about orienting graphs

In this section we consider the following question:

Question 1. *Given a graph G and a non-negative integer k, does there exist an orientation of G so that the obtained digraph is weak-odd k-edge-colorable?*

In the light of Corollary 1, the only two nontrivial cases arise when: $k = 1$ or $k = 2$.

3.1. Weak-odd 2-edge-colorability.

In this subsection, we provide an affirmative answer to Question 1 in the case $k = 2$. For that purpose, we will need the following sufficient condition for orientability in case $k = 1$.

Lemma 2. *Every forest F can be oriented so that the obtained digraph D is weak-odd 1-edge-colorable.*

Proof. We intend to orient the edges from $E(F)$ so that for every vertex v of the obtained digraph D, both $d_D^+(v)$ and $d_D^-(v)$ are odd or zero. We may assume F is a non-trivial rooted tree, say r is the root. Orient first the edges from $E_F(r)$ as follows: if $d_F(r)$ is odd, then make r a source; on the other hand, if $d_F(r)$ is even, then make the indegree of r equal to 1. Look for a non-pendant vertex $v \neq r$ for which just one edge $e \in E_F(v)$ is oriented so far. Since F is connected and acyclic, as long as $E(F)$ is not fully oriented, there exists such a vertex v. The partial orientation of $E_F(v)$ can always be completed so that both the outdegree and indegree of v are odd or zero. Namely, if $d_F(v)$ is odd then orient the rest of $E_F(v)$ consistently with the orientation of e (thus v becomes a peripheral vertex); on the other hand, if $d_F(v)$ is even, then orient the rest of $E_F(v)$ uniformly but inconsistently with e. Hence, repetition of the described procedure gives a required orientation of F. □

The stated condition is far from being necessary, but we can use it to settle the case $k = 2$ of Question 1.

Theorem 3. *Every graph G can be oriented to obtain a digraph D admitting a weak-odd edge-coloring using the color set $\{1, 2\}$ with every pendant edge colored by* 1.

Proof. Suppose the opposite, and let G be a counter-example which minimizes $|E(G)|$. By Lemma 2, there must be a cycle C in G. Orient the edges of C consistently (i.e. turn C into a directed cycle). Consider its edge-complement $\hat{C} = G - E(C)$. Modify the graph $\hat{C}$ by splitting into $d_{\hat{C}}(v)$ pendant vertices every $v \in V(C)$ of degree $d_{\hat{C}}(v) \geq 2$. After this is done, let G' be the graph obtained from $\hat{C}$. Since G' is of smaller size than G, it can be oriented so that the obtained digraph D' admits a weak-odd edge-coloring φ' using the color set $\{1, 2\}$ with every pendant edge colored by 1. Combined with the already given orientation of C, this gives us an orientation of G, and simultaneously defines an edge-coloring φ of the obtained digraph D as described in the statement (the color 2 is odd at every $v \in V(C)$). This contradicts the choice of G and completes the proof. □

3.2. **Weak-odd 1-edge-colorability.** In this subsection, we study the case $k = 1$ of Question 1. For simplicity of presentation, we often refer to any orientation of G for which the respective digraph D is weak-odd 1-edge-colorable as a *good orientation* of G. Moreover, the notation for the outdegree (resp. indegree) of a vertex $v \in V(D)$ is simplified to $d^+(v)$ (resp. $d^-(v)$). The condition for the respective digraph is that every vertex v has both $d^+(v)$, $d^-(v)$ odd or zero. Note that, unlike the case $k = 2$ of Problem 1, there exist graphs which do not admit good orientations (see Fig. 2).

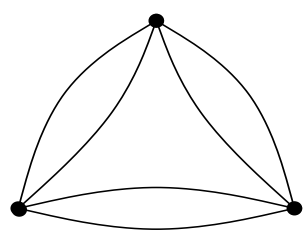

FIGURE 2. Example of a graph not admitting a good orientation.

For an arbitrary graph G, and let us denote by $\mathcal{O}_G$ (resp. $\mathcal{E}_G$) the set of its odd (resp. even) vertices. For the particular case when $\mathcal{O}_G = \emptyset$, i.e. of an even graph G, a complete characterization in terms of the existence of a good orientation is possible.

Theorem 4. *For any even connected non-empty graph G, the following four statements are equivalent:*

- (i) *For some orientation of G, the obtained digraph is weak-odd 1-edge-colorable.*
- (ii) *For some orientation of G, the set $S = \{v \in V(G) : d^+(v), d^-(v)$ are even$\}$ is even-sized.*
- (iii) *For any orientation of G, the set $S = \{v \in V(G) : d^+(v), d^-(v)$ are even$\}$ is even-sized.*
- (iv) *The set $T = \{v \in V(G) : d(v)$ is divisible by $4\}$ is even-sized.*

Proof. The implication $(i) \Rightarrow (ii)$ is obvious, since a particular orientation of G is good if and only if in the corresponding digraph D the set S is empty. For the opposite implication, consider an orientation of G for which the set S (in the obtained digraph D) is even-sized. We may assume that S is non-empty, say $S = \{u_1, u_2, \ldots, u_{2k}\}$. For $i = 1, \ldots, k$, select an arbitrary undirected $u_{2i-1}u_{2i}$-path P_i in D, and reverse the orientation of every edge of P_i. By doing this, we obtain a digraph for which the corresponding S is empty. This proves the implication $(ii) \Rightarrow (i)$.

Next, we demonstrate that given an arbitrary orientation of G, the parity of $|S|$ is an invariant of G, i.e. it is independent from the given orientation. Namely, select an arbitrary edge $e \in E(G)$ and reverse its orientation. Then, the size of the set S either increases by 2, stays the same, or decreases by 2. Hence, the parity of $|S|$ remains unaltered, which proves the equivalence $(ii) \Leftrightarrow (iii)$.

Consider now an 'Eulerian orientation' of G, i.e. take an Eulerian tour of G and orient every traversed edge accordingly. We thus obtain a regular digraph D, i.e. for every vertex $v \in V(D)$ holds $d^+(v) = d^-(v)$. Hence, the set T (defined in (iv)) satisfies $T = \{v \in V(G) : d^+(v), d^-(v)$ are even$\}$. Since $(ii) \Leftrightarrow (iii)$, this yields $(ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$.

□

We turn next to the more general case when $\mathcal{O}_G \neq \emptyset$. We show that it is always possible to decide in polynomial time whether the graph G admits a good orientation. First we give a necessary condition for the existence of a good orientation. Namely, it is straightforward that whenever G admits one, we can orient the edges from the set

$$E(\mathcal{O}_G) = \bigcup_{u \in \mathcal{O}_G} E(u)\,,$$

so that every odd vertex u becomes peripheral (a source or sink). We refer to any such orientation of $E(\mathcal{O}_G)$ as a *proper partial orientation* of G. It is a simple matter to characterize the existence of a proper partial orientation.

Proposition 1. *The following two statements are equivalent:*

(i) *G admits a proper partial orientation.*
(ii) *The induced subgraph $G[\mathcal{O}_G]$ is bipartite.*

Proof. Suppose that (i) holds. Then, there cannot be any odd cycle C (a cycle of odd length) in G such that $V(C) \subseteq \mathcal{O}_G$. Namely, if such a C exists, then on an arbitrary traversing of C, the passed vertices are alternately sources and sinks. But this contradicts its odd length.

On the other hand, if (ii) holds then we may consider a pair X, Y of partite sets for $G[\mathcal{O}_G]$. Clearly, there exists a proper partial orientation of G for which every $u \in X$ (resp. $u \in Y$) becomes a source (resp. sink). □

Hence, a necessary condition for the existence of a good orientation of G is that the subgraph $G[\mathcal{O}_G]$ is bipartite. In the particular case of an odd graph G, this necessary condition also suffices, i.e. we deduce the following.

Corollary 2. *For any odd graph G, the next two statements are equivalent:*

(i) *For some orientation of G, the obtained digraph is weak-odd* 1*-edge-colorable.*
(ii) *G is bipartite.*

In the remainder of this section, we restrict to connected graphs G that are neither even nor odd (i.e. $\emptyset \subset \mathcal{O}_G, \mathcal{E}_G \subset V(G)$), and for which $G[\mathcal{O}_G]$ is bipartite. For such a G, we shall describe a simple procedure for deciding whether it admits a good orientation.

We 'decompose' G into *patches* of two particular kinds and introduce special names for them. This is achieved by 'breaking in halves' each $e \in [\mathcal{O}_G, \mathcal{E}_G]$, i.e. every edge e having endpoints of different parities (one even and one odd). By doing this, we introduce a *half-edge* at each of its endpoints. Observe that, after all these 'breakages in halves' are completed, we are left with partitions of the sets $\mathcal{O}_G$ and $\mathcal{E}_G$ into 'connected' subsets $\mathcal{O}_1, \dots, \mathcal{O}_t$ and $\mathcal{E}_1, \dots, \mathcal{E}_s$, respectively. For $j = 1, \dots, t$, the subgraph $G[\mathcal{O}_j]$ along with the set of half-edges incident to $\mathcal{O}_j$ is called the *j-th odd patch* of G, denoted by X_j. Similarly, for $i = 1, \dots, s$, the subgraph $G[\mathcal{E}_i]$ along with the set of half-edges incident to $\mathcal{E}_i$ is the *i-th even patch* of G, denoted by Y_i. Let a_{ij} be the number of half-edges in Y_i whose other halves belong to X_j.

Now, take an arbitrary proper partial orientation of G. This induces a (complete) orientation of every odd patch, whereas only the half-edges of every even patch are oriented. Complete the orientation of G by choosing an arbitrary direction for every edge belonging to an even patch, and call it a *trial orientation* of G. In the obtained digraph, consider the sets $S_i = \{v \in \mathcal{E}_i : d^+(v), d^-(v) \text{ are even}\}$, for $i = 1, \dots, s$, and let $b_i = |S_i|$.

Observe that, by reversing the direction of any edge in Y_i the parity of b_i stays unaltered. On the other hand, by reversing the direction of any half-edge in Y_i the parity of b_i changes.

Theorem 5. *The next two statements are equivalent:*

(i) *The trial orientation of G can be modified so that the obtained digraph is weak-odd 1-edge-colorable.*

(ii) *The following system of linear equations is solvable over the prime field $\mathbb{Z}_2$.*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1t}x_t = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2t}x_t = b_2 \\ \qquad\qquad \cdots\cdots\cdots \\ a_{s1}x_1 + a_{s2}x_2 + \ldots + a_{st}x_t = b_s \end{cases}$$

Proof. Suppose (i) holds, i.e. at least one good orientation of G exists. Compare one such with the trial orientation. On any odd patch X_j, either the two orientations completely match or are completely opposite. Define $\mathbf{x}_j$ to be equal to 0 (resp. 1) whenever the two orientation are the same (resp. opposite) on the edges of X_j. Then clearly $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_t$ is a solution of the above system of linear equations over the prime field $\mathbb{Z}_2$.

For the other direction of the stated equivalence, suppose (ii) holds, and let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_t$ be a solution of the above system of linear equations over the prime field $\mathbb{Z}_2$. Modify the trial orientation in accordance with the following: for every j with $\mathbf{x}_j = 1$, reverse the directions of all the edges of G incident to $\mathcal{O}_j$. We thus obtain an orientation of G such that the current set $S_i = \{v \in \mathcal{E}_i : d^+(v), d^-(v) \text{ are even}\}$ is even-sized, for $i = 1, \ldots, s$. On every even patch Y_i we employ the path-reversal trick from the proof of Theorem 4. We may assume that S_i is non-empty, say $S_i = \{u_1^{(i)}, u_2^{(i)}, \ldots, u_{2k}^{(i)}\}$. For $r = 1, \ldots, k$, select an arbitrary undirected $u_{2r-1}^{(i)}u_{2r}^{(i)}$-path P_r in $G[\mathcal{E}_i]$, and reverse the orientation of every edge of P_r. After having done this 'inside' every Y_i, we end up with a good orientation of G. □

Consequently, we are able to decide (in polynomial time) whether G admits a good orientation, and if this is the case we have a way of modifying the trial orientation into a good one.

Example. It was already shown in Lemma 2 that every forest admits a good orientation. Let us give an alternative argument for this fact in the spirit of the described procedure. It is enough to consider a non-trivial tree T which is not an odd graph. Since T is acyclic, the induced subgraph $T[\mathcal{O}_T]$ is clearly bipartite. Take an arbitrary proper partial orientation of

T and extend it to a trial orientation. Denote by A and B the respective matrix and augmented matrix of the corresponding system of linear equations over the prime field $\mathbb{Z}_2$. By Kronecker-Capelli theorem (see [2]), a system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1t}x_t = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2t}x_t = b_2 \\ \quad\quad\quad\quad \cdots\cdots\cdots \\ a_{s1}x_1 + a_{s2}x_2 + \ldots + a_{st}x_t = b_s \end{cases}$$

is compatible (i.e. solvable) over a given field if and only if the rank of the coefficient matrix $A = \|a_{ij}\|$ is equal to the rank of the augmented matrix B obtained from A by adding the column of free coefficients b_i. Therefore, we are left to explain why does $rank(A)$ equal $rank(B)$ under the given circumstances.

Since T is acyclic, every $a_{ij} \in \{0,1\}$. We claim that: (1) $t > s$; and (2) the rows of A are linearly independent over $\mathbb{Z}_2$. Note that (1) and (2) together imply $rank(A) = s = rank(B)$. In order to prove these two claims, for $j = 1,\ldots,t$ and $i = 1,\ldots,s$, contract $\mathcal{O}_j$ into a vertex $\mathcal{O}'_j$ and $\mathcal{E}_i$ into a vertex $\mathcal{E}'_i$. The obtained graph T' is again a tree of order $s+t$. Moreover, both $\{\mathcal{O}'_1,\ldots,\mathcal{O}'_t\}$ and $\{\mathcal{E}'_1,\ldots,\mathcal{E}'_s\}$ are independent sets of vertices in T'. Observe that every leaf of T' is an $\mathcal{O}'_j$, and every $\mathcal{E}'_i$ is an even vertex of T'. The first part of this observation is obvious, and the second follows from the 'handshake lemma'. These two facts about the tree T' easily yield claim (1): it is enough to consider T' as a rooted tree with $\mathcal{E}'_1$ as a root, and then count the vertices on odd (resp.) even distance from the root. To demonstrate claim (2), let $\mathcal{O}'_j$ be a leaf of T' and $\mathcal{E}'_i$ be its only neighbor. Then, only the i-th entry is non-zero in the j-th column of A, which clearly implies (2).

Acknowledgements. This work is partially supported by ARRS Program P1-0383 and by Creative Core FISNM-3330-13-500033.

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СЛАБО-НЕПАРНО РЕБРЕНО-БОЕЊЕ НА ДИГРАФОВИ

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Р е з и м е

Слабо-непарно ребрено-боење на диграф D претставува (не задолжително правилно) ребрено-боење при кое кај секое теме v барем една боја c го задоволува следниов услов: ако $d^+(v) > 0$ тогаш c се појавува непарен број пати на излезните ребра кај v; ако $d^-(v) > 0$ тогаш c се појавува непарен број пати на влезните ребра кај v. Минималниот број бои доволни за слабо-непарно ребрено-боење на D е негов слабо-непарен хроматски индекс, со ознака $\chi'_{wo}(D)$. Во оваа статија докажуваме дека за секој диграф D важи $\chi'_{wo}(D) \leq 3$, при што равенство е достижно. Проучуваме кога даден граф дозволува ориентација при која добиениот диграф е слабо-непарно 1-ребрено-обојлив. Покажуваме и дека секој граф дозволува ориентација при која добиениот диграф е слабо-непарно 2-ребрено-обојлив.

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