2. Solution of the problem

In order to find the eigenvalues of the matrix A, we introduce the following matrices:

$M = [m_{ij}]$ with elements

$$m_{ii} = i, \qquad (0 \le i \le n)$$

$$m_{i,i+1} = -(i+1), \qquad (0 \le i \le n-1)$$

$$m_{ij} = 0, \qquad \text{(otherwise)}$$

$N = [n_{ij}]$ with elements

$$n_{ii} = n - i, \qquad (0 \le i \le n)$$

$$n_{i,i-1} = -(n+1-i), \qquad (1 \le i \le n)$$

$$n_{ij} = 0, \qquad \text{(otherwise)}$$

$T = [t_{ij}]$ with elements

$$t_{ii} = i, \qquad (0 \le i \le n)$$

$$t_{i,i-1} = -(n+1-i), \qquad (1 \le i \le n)$$

$$t_{ij} = 0, \qquad \text{(otherwise)}$$

$D = diag(0, 1, 2, \cdots, n)$,

$B = [b_{ij}] = \left[\binom{j}{i}\right], \qquad (0 \le i, j \le n)$

where $\binom{j}{i} = 0$ for $j < i$.

Now we prove that $BN = TB$. The (i,j)-th element of the matrix BN is

$$\binom{j}{i}(n-j) - \binom{j+1}{i}(n-j) = -(n-j)\binom{j}{i-1},$$

and the (i,j)-th element of the matrix TB is

$$i\binom{j}{i} - (n+1-i)\binom{j}{i-1} = -(n-j)\binom{j}{i-1}.$$

We prove also that $BM = DB$. Indeed, the (i,j)-th element of the matrix BM is

$$\binom{j}{i}j - \binom{j-1}{i}j = \binom{j-1}{i-1}j = i\binom{j}{i},$$

and it is obviously equal to the (i,j)-th element of the matrix DB.